UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 14 June, 2021 12:00 pm to 3:00 pm

PAPER 326

INVERSE PROBLEMS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Spectral Regularisation

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $A\colon \mathcal{X} \to \mathcal{Y}$ a linear bounded operator. We consider the following inverse problem

$$Au = f$$

for $f \in \mathcal{Y}$.

1. Give the definition of a *well-posed problem* and an *ill-posed problem* in the sense of Hadamard. Give the definitions of a *regularisation* and a *convergent regularisation.*

2. Let (σ_i, x_i, y_i) be the singular system of a compact operator A. Consider the following operator
$$T_n := AP_n,$$
where P_n is the projector onto the subspace spanned by the first n singular vectors x_i, $i = 1, ..., n$. Show that the Moore-Penrose inverse of T_n is given by
$$(T_n)^\dagger = A^\dagger Q_n,$$
where Q_n is the projector onto the subspace spanned by the first n singular vectors y_i, $i = 1, ..., n$, and $A^\dagger$ is the Moore-Penrose inverse of A.

   Hint: verify the Moore-Penrose equations.

3. The following iterative process is referred to as *Landweber iteration*
$$\begin{cases} u^{k+1} = (I - \tau A^* A)u^k + \tau A^* f, \quad k = 1, 2, ..., \\ u^0 \equiv 0, \end{cases}$$
where $f \in \mathcal{D}(A^\dagger)$ and $\tau > 0$.

   (a) Show that this method can be expressed as a spectral regularisation method and give the corresponding representation in terms of the singular system of A.

   (b) What is the regularisation parameter in this method?

   (c) Give sufficient conditions under which Landweber iteration defines a regularisation [proof required]. Write these conditions in terms of the norm of A.

   Hint: you can use the fact that for any $k \in \mathbb{N}$ and any $a > 0$
$$\sum_{i=1}^{k} (1-a)^{k-i} = \frac{1}{a}(1 - (1-a)^k).$$

2 Variational Regularisation

1. Define *Total Variation* of a function $u \in L^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Define the *spaces* $\mathrm{BV}(\Omega)$ *and* $\mathrm{BV}_0(\Omega)$. Write down the *Poincaré inequality* for Total Variation.

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces, $A\colon \mathcal{X} \to \mathcal{Y}$ a linear bounded operator and $\mathcal{J}\colon \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ a proper convex lower semicontinuous functional. Give the definition of a $\mathcal{J}$*-minimising solution* $u^\dagger_{\mathcal{J}}$ of the equation

$$Au = f$$

with $f \in \mathcal{R}(A)$. Formulate the *source condition* for a $\mathcal{J}$-minimising solution $u^\dagger_{\mathcal{J}}$. Prove that the source condition is equivalent to the following *range condition*

Range condition. For any fixed $\alpha > 0$ there exists $g \in \mathcal{Y}$ such that

$$u^\dagger_{\mathcal{J}} \in \arg\min_{u \in \mathcal{X}} \frac{1}{2}\|Au - g\|^2_{\mathcal{Y}} + \alpha \mathcal{J}(u).$$

Hint: Consider necessary and sufficient first order optimality conditions for the optimisation problem above.

3. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{J}\colon \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ a proper convex lower semicontinuous and absolutely one-homogeneous functional.

(a) Show that for any $c > 0$ and any $u \in \mathcal{X}$

$$\partial \mathcal{J}(cu) = \partial \mathcal{J}(u),$$

where $\partial \mathcal{J}$ is the subdifferential of $\mathcal{J}$.

Hint: You may use the characterisation of the subdifferential of an absolutely one-homogeneous functional from one of the example sheets.

(b) A function $f \in \mathcal{X}$ is called an eigenfunction of $\mathcal{J}$ corresponding to the eigenvalue $\lambda \in \mathbb{R}$ if

$$\lambda f \in \partial \mathcal{J}(f).$$

Show that if f is an eigenfunction corresponding to an eigenvalue $\lambda > 0$ then the function $u = (1 - \lambda\alpha) f$ solves the following optimisation problem

$$\min_{u \in \mathcal{X}} \frac{1}{2}\|u - f\|^2 + \alpha \mathcal{J}(u),$$

where $\alpha < \frac{1}{\lambda}$.

Hint: Consider necessary and sufficient first order optimality conditions for the optimisation problem above.

3 Bayesian inverse problems and well-posedness

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{X}$ be a separable $\mathbb{R}$-Banach space and $\mathcal{BX}$ be the associated Borel-σ-algebra. Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the positive integers. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces and let $g : \Omega_1 \to \Omega_2$ be a function. We denote $g : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$, if g is measurable from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$.

1. (a) Define a *Bayesian inverse problem* on $(\mathcal{X}, \mathcal{BX})$ with prior $\mu_0 \in \mathrm{Prob}(\mathcal{X}, \mathcal{BX})$ and likelihood $L : (\mathcal{X} \times \mathcal{Y}, \mathcal{BX} \otimes \mathcal{BY}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$.

   (b) Define the *total variation distance* d_{TV} on $\mathrm{Prob}(\mathcal{X}, \mathcal{BX})$.

   (c) Define the $(\mathrm{Prob}(\mathcal{X}, \mathcal{BX}), d_{\mathrm{TV}})$-*well-posedness of a Bayesian inverse problem* and explain the role of d_{TV} in this definition.

   (d) Give the four general assumptions on μ_0 and L under which the associated Bayesian inverse problem is $(\mathrm{Prob}(\mathcal{X}, \mathcal{BX}), d_{\mathrm{TV}})$-well-posed (a proof is not necessary).

2. Let $\mathcal{Y} := \mathbb{R}^k$, where $k \in \mathbb{N}$. We consider the inverse problem where we aim to identify $u \in \mathcal{X}$ from the observation $f_n \in \mathcal{Y}$ which is defined as a sample from

$$\mathcal{A}(u) + N,$$

   where $\mathcal{A} : (\mathcal{X}, \mathcal{BX}) \to (\mathcal{Y}, \mathcal{BY})$ and $N : (\Omega, \mathcal{F}) \to (\mathcal{Y}, \mathcal{BY})$, where $N \sim \mu_{\mathrm{noise}}$. Here, μ_{noise} is the probability measure which has the following density with respect to the Lebesgue measue λ_k:

$$g(y) = \exp(-2\|y\|_1) \qquad (y \in \mathcal{Y}).$$

   (a) Show that for any value of the parameter $u \in \mathcal{X}$, the probability distribution of the observable $\mathbb{P}(\mathcal{A}(u) + N \in \cdot)$ has a λ_k-density. Write down a likelihood function that is associated to the inverse problem of identifying u from the observation f_n and show that it is measurable.

   (b) Let $\mu_0 \in \mathrm{Prob}(\mathcal{X}, \mathcal{BX})$ be a probability measure. Find a likelihood L' that is $\mu_0 \otimes \lambda_k$-almost everywhere identical to the likelihood in (a) such that the Bayesian inverse problem with prior μ_0 and likelihood L' is $(\mathrm{Prob}(\mathcal{X}, \mathcal{BX}, \mu_0), d_{\mathrm{TV}})$-well-posed.

   Hint: You may apply the conditions in part 1(d) without proof.

[QUESTION CONTINUES ON THE NEXT PAGE]

3. (a) Let $\mu, \mu' \in \mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X})$. Show that there is a σ-finite measure ν on $(\mathcal{X}, \mathcal{B}\mathcal{X})$, such that $\mu, \mu' \ll \nu$.

(b) Let now ρ be some σ-finite measure on $(\mathcal{X}, \mathcal{B}\mathcal{X})$ and

$$\mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X}, \rho) := \{\mu \in \mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X}) : \mu \ll \rho\}.$$

We define the Hellinger distance between $\mu, \mu' \in \mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X}, \rho)$ by

$$d_{\mathrm{Hel}}(\mu, \mu') = \sqrt{\int_{\mathcal{X}} \left(\sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\rho}} - \sqrt{\frac{\mathrm{d}\mu'}{\mathrm{d}\rho}} \right)^2 \mathrm{d}\rho}$$

Show that this function is well-defined by proving that the integral

$$\int_{\mathcal{X}} \left(\sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\rho}} - \sqrt{\frac{\mathrm{d}\mu'}{\mathrm{d}\rho}} \right)^2 \mathrm{d}\rho$$

is finite.

(c) Let $\mu, \mu' \in \mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X}, \rho)$. Show that the following inequality holds:

$$d_{\mathrm{Hel}}(\mu, \mu')^2 \leqslant 2 d_{\mathrm{TV}}(\mu, \mu')$$

Hint: You may use the formula $d_{\mathrm{TV}}(\mu, \mu') = \frac{1}{2} \int_{\mathcal{X}} \left| \frac{\mathrm{d}\mu}{\mathrm{d}\rho} - \frac{\mathrm{d}\mu'}{\mathrm{d}\rho} \right| \mathrm{d}\rho$ *without proving it.*

(d) Consider a Bayesian inverse problem with prior $\mu_0 \in \mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X})$ and likelihood $L : (\mathcal{X} \times \mathcal{Y}, \mathcal{B}\mathcal{X} \otimes \mathcal{B}\mathcal{Y}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$ and assume that it is $(\mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X}), d_{\mathrm{TV}})$-well-posed. Show that the Bayesian inverse problem is then also $(\mathrm{Prob}(\mathcal{X}, \mathcal{B}\mathcal{X}, \mu_0), d_{\mathrm{Hel}})$-well-posed.

4 Gaussian measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{X}$ be a separable $\mathbb{R}$-Banach space and $\mathcal{B}\mathcal{X}$ be the associated Borel-σ-algebra. Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the positive integers. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces and let $g : \Omega_1 \to \Omega_2$ be a function. We denote $g : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$, if g is measurable from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$.

Hint: Throughout this exercise you may use the following results without proof: Let $m_1, m_2, \sigma_1^2, \sigma_2^2, a, b \in \mathbb{R}$, and $\sigma_1^2, \sigma_2^2 \geqslant 0$. Moreover, let $\xi_1 : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$ and $\xi_2 : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$ be random variables, where $\xi_1 \sim \mathrm{N}(m_1, \sigma_1^2), \xi_2 \sim \mathrm{N}(m_2, \sigma_2^2)$ are independent and identically distributed. Then, (i) $a\xi_1 + b\xi_2 \sim \mathrm{N}(am_1 + bm_2, a^2\sigma_1^2 + b^2\sigma_2^2)$ and (ii) $\int_\Omega \xi_1 \mathrm{d}\mathbb{P} = m_1$ and $\int_\Omega \xi_1^2 \mathrm{d}\mathbb{P} = \sigma_1^2 + m_1^2$.

1. Give the definition of a *Gaussian measure on* $(\mathcal{X}, \mathcal{B}\mathcal{X})$ and its *mean and covariance operator*.

2. Let $\mathcal{X} := L^2([0,1], \mathcal{B}[0,1], \lambda_1)$ and $A \in \mathcal{B}[0,1]$, with $\lambda_1(A) \in (0,1)$. Moreover, let $\varphi_1 := \mathbf{1}_A$, and $\varphi_2 := \mathbf{1}_{[0,1]\setminus A}$. Let $\xi_1 : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$ and $\xi_2 : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$ be random variables with $\xi_1, \xi_2 \sim \mathrm{N}(0, 1^2)$ independent and identically distributed.

   (a) Show that $\varphi_1, \varphi_2 \in \mathcal{X}$.

   (b) Show that the distribution of the random variable $U : (\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}\mathcal{X})$ given by
   $$U := \xi_1\varphi_1 + \xi_2\varphi_2$$
   is a Gaussian measure on $(\mathcal{X}, \mathcal{B}\mathcal{X})$ and determine its mean and covariance operator.

   (c) Show that φ_1, φ_2 are orthogonal eigenvectors of the covariance operator of U. Then determine the associated eigenvalues and show how they depend on the Lebesgue measure of A.

3. Let $D := [0,1]^n, n \in \mathbb{N}$ be a compact space and $\mathcal{X} := L^2(D, \mathcal{B}D, \lambda_n)$. Moreover, let $k \in \mathbb{N}$, $(\varphi_i)_{i=1}^k \in \mathcal{X}^k$ be an orthonormal family, and $\nu_1, \ldots, \nu_k > 0$. Let $\xi : (\Omega, \mathcal{F}) \to (\mathbb{R}^k, \mathcal{B}\mathbb{R}^k)$ be a random variable, with $\xi_1, \ldots, \xi_k \sim \mathrm{N}(0, 1^2)$ independent and identically distributed and $U : (\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}\mathcal{X})$ be a Gaussian random field, given by $U := \sum_{i=1}^k \sqrt{\nu_i}\xi_i\varphi_i$. Show that the random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}\mathbb{R})$,
   $$\omega \mapsto \int_D U(\omega; x)\mathrm{d}\lambda_n(x)$$
   is distributed according to a Gaussian measure and determine its mean and variance. Finally, determine $\int_\Omega \|U\|_{\mathcal{X}}^2 \mathrm{d}\mathbb{P}$.

END OF PAPER