MATHEMATICAL TRIPOS Part III

Wednesday, 16 June, 2021 12:00 pm to 3:00 pm

PAPER 323

QUANTUM INFORMATION THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **FOUR** questions.*
*There are **FIVE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)

(i) State and prove the Schmidt Decomposition Theorem for a bipartite pure state $|\phi_{AB}\rangle$. Define the Schmidt rank.

(ii) For a quantum channel $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and $\rho \in \mathcal{D}(\mathcal{H})$, define the entanglement fidelity $F_e(\rho, \Lambda)$. Using Schmidt Decomposition Theorem or otherwise show that $F_e(\rho, \Lambda) = \sum_{k=1}^{n} |\mathrm{Tr}(A_k \rho)|^2$, where $\Lambda(\rho) = \sum_{k=1}^{n} A_k \rho A_k^\dagger$, and Kraus operators $\{A_k\}_{k=1}^{n}$.

(iii) Compute $F_e(|+\rangle\langle+|, \mathcal{E}_{\frac{1}{2}})$, where $\mathcal{E}_{\frac{1}{2}} : \mathcal{D}(\mathcal{H}_2) \to \mathcal{D}(\mathcal{H}_3)$, $\mathcal{E}_{\frac{1}{2}}(\rho) = \frac{1}{2}\rho + \frac{1}{2}|e\rangle\langle e|$, $|e\rangle = (0,0,1)^T$, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $\rho \in \mathcal{D}(\mathcal{H}_2)$, $\dim(\mathcal{H}_k) = k$.

(b) For a density matrix $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, define a Schmidt number $\mathrm{SN}(\rho)$ to be the least number s such that ρ can be represented as $\rho = \sum_{k=1}^{n} |v_k\rangle\langle v_k|$ where all bipartite states $|v_k\rangle$ have Schmidt rank at most s.

(i) Compute $\mathrm{SN}(\rho)$ where:

1) $\rho = \sum_{i=1}^{2} p_i \rho_i \otimes \sigma_i$, where $\rho_i, \sigma_i \in \mathcal{D}(\mathcal{H})$, $\dim(\mathcal{H}) = 2$, $p_1 + p_2 = 1, p_1, p_2 \geqslant 0$

2) $\rho = |\Phi^+\rangle\langle\Phi^+|$

3) $\rho = \frac{1}{4}(|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|)$

where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$

(ii) Fix $k \in \mathbb{N}$, and consider the map $\Lambda_k : \mathcal{B}(\mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_n)$, $\dim(\mathcal{H}_n) = n$ which acts as $\Lambda_k(\rho) = k\mathbb{1} - \rho$. Show that for $\sigma \in \mathcal{D}(\mathcal{H}_n \otimes \mathcal{H}_n)$

$$\mathrm{SN}(\sigma) \leqslant k \implies (id_n \otimes \Lambda_k)(\sigma) \geqslant 0$$

2

(a)

(i) Consider $|\Psi\rangle_{AR}$, $|\Phi\rangle_{AR}$ to be distinct purifications of $\rho_A \in \mathcal{D}(\mathcal{H}_A)$. Show that $|\Psi\rangle_{AR} = (\mathbb{1}_A \otimes U_R)|\Phi\rangle_{AR}$.

(ii) Suppose $\rho_A = \sum_{i=1}^n \lambda_i \rho_A^i$, where $\rho_A^i = |a_i\rangle\langle a_i|$, $|a_i\rangle \in \mathcal{H}_A$, $\dim(\mathcal{H}_A) = n$, $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \geqslant 0$ for $i = 1 \dots n$.

1) Show that $|\Psi\rangle_{AR} = \sum_{i=1}^n \sqrt{\lambda_i}|a_i\rangle_A|b_i\rangle_R$ is a purification of ρ_A, where $\{|b_i\rangle\}_{i=1}^n$ is an orthonormal basis of $\mathcal{H}_R$, and $\dim\mathcal{H}_R = \dim\mathcal{H}_A$.

2) Show that $|\Psi\rangle_{RA} = \sqrt{n}(\mathbb{1}_R \otimes \sqrt{\rho_A})|\Omega\rangle_{RA}$ is a purification of ρ_A, where $\sqrt{\rho_A}$ is a unique positive square root of ρ_A, and $|\Omega\rangle_{RA} = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle_R|i\rangle_A$

3) Consider the following state: $\rho_{X_1X_2A_1A_2} = \sum_{i=1}^n \sum_{j=1, j\neq i}^n p_{ij}|i\rangle\langle i|_{X_1} \otimes |j\rangle\langle j|_{X_2} \otimes \rho_{A_1}^i \otimes \rho_{A_2}^j$, $\sum_{i=1}^n p_{ij} = 1$, $p_{ij} \geqslant 0$ for $i = 1 \dots n$. . Write down a purification $|\Psi_{X_1X_2A_1A_2R}\rangle$ of $\rho_{X_1X_2A_1A_2}$. What is the smallest dimension of $\mathcal{H}_R$ required to purify states of this form?

(b)

(i) Suppose that $\rho_1 = \sum_{k=1}^n p_k|\phi_k\rangle\langle\phi_k|$, $\rho_2 = \sum_{k=1}^n q_k|\psi_k\rangle\langle\psi_k|$. Show that $\rho_1 = \rho_2$ if and only if $\sqrt{q_k}|\psi_k\rangle = \sum_{j=1}^n U_{kj}\sqrt{p_j}|\phi_j\rangle$, for some unitary matrix $U = \{U_{kj}\}_{k,j=1}^n$

(ii) Consider two quantum states ρ_A, σ_A where $F(\rho_A, \sigma_A) \geqslant 1 - \epsilon$, $\epsilon \in (0, 1)$. Let $|\Psi_{AR}^\rho\rangle$ be a purification of ρ_A. Can we find $|\Phi_{AR}^\sigma\rangle$ such that $F(|\Psi_{AR}^\rho\rangle, |\Phi_{AR}^\sigma\rangle) \geqslant 1 - \epsilon$? Clearly state any theorem that you use.

3

a)

(i) State the Stinespring Dilation Theorem for a quantum channel $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$.

(ii) Consider a qubit phase flip channel $\Lambda_p(\rho) = (1-p)\rho + p\sigma_z\rho\sigma_z$.

1) Let $\rho = \frac{1}{2}(\mathbb{1} + \bar{r}\cdot\hat{\sigma})$, where $\bar{r} \in \mathbb{R}^3$, $|\bar{r}| = 1$, and a vector of Pauli matrices $\hat{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$. Find the action of Λ_p on $\bar{r}$.

2) Consider $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Express the output of the channel $\tilde{\rho} = \Lambda_{1/2}(\rho)$ in the computational basis.

3) In the experiment the state ρ is measured in the computational basis, but due to malfunction all information about the measurement outcome has been lost. Describe the post-measurement state σ in this case and compare it with $\Lambda_{1/2}(\rho)$.

b) Define a Pauli channel $\Lambda(\rho) = \sum_{i,j=0}^{1} p_{i,j}\sigma_z^i\sigma_x^j\rho\sigma_x^j\sigma_z^i$, where $\{p_{i,j}\}_{i,j=0}^1$ is a probability distribution.

(i) Show that Λ can be written as

$$\Lambda(\rho) = p_0\rho + \sum_{i\in\{x,y,z\}} p_i\sigma_i\rho\sigma_i,$$

and express p_0, p_x, p_y, p_z in terms of $\{p_{i,j}\}_{i,j=0}^1$.

(ii) Let $\rho = \frac{1}{2}(\mathbb{1} + \bar{r}\cdot\hat{\sigma})$ as above. Find the action of Λ on $\bar{r}$.

4

(a)

(i) Define the trace distance $D(\rho,\sigma)$ and the fidelity $F(\rho,\sigma)$ between two quantum states $\rho,\sigma \in \mathcal{D}(\mathcal{H})$.

(ii) Prove that $D(\rho,\sigma)$ can be expressed as

$$D(\rho,\sigma) = \max_{0 \leqslant P \leqslant \mathbf{1}} \mathrm{Tr}(P(\rho - \sigma)).$$

(iii) Compute $D(|\Phi^+\rangle, |\Psi^-\rangle)$ and $F(|\Psi^+\rangle, |\Phi^-\rangle)$, where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, and $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

(iv) Suppose we know τ, where $\tau = \rho_1 - \rho_2$, where $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$, but we have no information about of ρ_1 and ρ_2. Does the knowledge of τ suffice to compute 1) $D(\rho_1,\rho_2)$ 2) $F(\rho_1,\rho_2)$?

(b) Consider the scenario of quantum binary hypothesis testing. Suppose Alice prepares a quantum system A in one of two states ρ_1 and ρ_2 with probabilities p_1 and p_2 respectively, and sends it to Bob through a noiseless quantum channel. Bob attempts to identify the state using binary POVM $\{E_1, \mathbb{1} - E_1\}$.

(i) Let $p_1 = p_2 = 1/2$. Show that the maximum probability of success is given by $p_{succ} = 1/2(1 + D(\rho_1,\rho_2))$.

(ii) Show that when ρ_1 is prepared with probability p_1, and ρ_2 is prepared with probability p_2 with $p_1 + p_2 = 1$, the maximum probability of success is given by $p_{succ} = 1/2(1 + ||p_1\rho_1 - p_2\rho_2||)$.

5

(a)

(i) State the Generalized Measurement Postulate.

(ii) Show that any generalized measurement can be implemented by introducing ancilla, allowing unitary dynamics and performing a projective measurement.

(b)

(i) Consider two states $\rho_1 = |0\rangle\langle 0|$, $\rho_2 = |\theta\rangle\langle\theta|$, where $|\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$, $\theta \in (0, \pi/2)$.

Construct a POVM that distinguishes ρ_1 and ρ_2 that does not make an error of misindentification.

(ii) Suppose Alice holds one of two states $|\phi_1\rangle_{A_1A_2} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, $|\phi_2\rangle_{A_1A_2} = b|00\rangle + a|01\rangle + d|10\rangle + c|11\rangle$, where $a, b, c, d \in (0, 1)$, $a < b < c < d$ and $\langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = 1$.

Alice sends the A_1 through the perfect quantum channel to Bob. Can Bob distinguish which state they share with Alice?

How will your answer change if Alice sends A_2 instead?

(c) Suppose a quantum state $\rho \in \mathcal{D}(\mathcal{H})$ is measured using a complete set of orthogonal projection operators $\{P_i\}$ but we never received the measurement outcomes. Describe the post-measurement state σ and show that $S(\sigma) \geqslant S(\rho)$. State the conditions when equality holds.

END OF PAPER