UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 24 June, 2021 12:00 pm to 2:00 pm

PAPER 322

BINARY STARS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 One common type of binary millisecond pulsar consists of a rapidly spinning neutron star of mass M_{ns} in a circular binary system with a low-mass white dwarf of mass M_{wd}. The orbital period P and the radial velocity amplitude K of the neutron star can be accurately measured. Deduce that we can define a mass function F such that

$$F = \frac{M_{\mathrm{wd}}^3 \sin^3 i}{(M_{\mathrm{ns}} + M_{\mathrm{wd}})^2} = \frac{P}{2\pi G} K^3, \qquad (*)$$

where i is the inclination of the system and G is Newton's gravitational constant, and show that $M_{\mathrm{wd}} > F$.

One model for the formation of such binary millisecond pulsars involves a low-mass red giant companion filling its Roche lobe and stably transferring mass to the neutron star. Explain briefly how this is expected to create the millisecond pulsar.

The luminosity L of the giant is a function of its core mass M_{c} only and its radius $R = f(L, M_{\mathrm{g}})$, where M_{g} is its total mass. Assume this relation holds until the giant loses its entire envelope at which point its radius rapidly shrinks. The giant's Roche lobe radius R_{L} is then given by

$$\frac{R_{\mathrm{L}}}{a} = \left(\frac{M_{\mathrm{g}}}{M_{\mathrm{g}} + M_{\mathrm{ns}}} \right)^{\frac{1}{3}},$$

where a is the semi-major axis of the orbit. Show that the orbital period of the binary millisecond pulsar is a function only of M_{wd}.

A large population of binary millisecond pulsars is observed. What is the expected distribution of inclinations $p(i)$, where $p(i)\,\mathrm{d}i$ is the probability that the inclination lies between i and $i+\mathrm{d}i$? The mass of a typical pulsar is found to be $M_{\mathrm{ns}} = 1.35\,M_{\odot}$. Assuming this is also true for millisecond pulsars, argue that the inclinations of their orbits can be found with $(*)$.

It is found that there is a distinct dearth of high inclination systems. Give two possible physical explanations of this.

2 Show that the orbital angular momentum of a circular binary system comprising two stars of masses M_1 and M_2, separation a and orbital angular velocity Ω is

$$J = \frac{M_1 M_2}{M} a^2 \Omega,$$

where $M = M_1 + M_2$. Assume that the orbit remains circular as the system evolves.

Star 1 is losing mass at a rate $-\dot{M}$ in an isotropic wind. Ignoring any spin angular momentum of the stars show that the orbital period P of the binary system and its total mass are related by

$$PM^2 = \text{const.}$$

Star 1 is a red giant the radius of which responds on a dynamical time-scale according to

$$R \propto M_1^{-n} \quad 0 < n < 1$$

while the radius of its Roche lobe $R_{\rm L}$ is approximated by

$$\frac{R_{\rm L}}{a} = 0.426 \left(\frac{M_1}{M} \right)^{\frac{1}{3}}.$$

Now suppose that the giant is exactly filling its Roche lobe and that wind mass loss continues on a time-scale much longer than any dynamical time-scale but also much shorter than the nuclear time-scale on which star 1 evolves. Show, by differentiating $\log R/R_{\rm L}$ or otherwise, that mass transfer is driven by the wind if

$$q = \frac{M_1}{M_2} < \frac{1+3n}{3(1-n)}. \qquad (\dagger)$$

What happens otherwise?

Show further that, when ($\dagger$) is satisfied, the rate of stable mass transfer to the main-sequence star is

$$\dot{M}_2 = -\frac{1+3n-3(1-n)q}{(1+q)(5-3n-6q)} \dot{M}.$$

What happens when $5 - 3n - 6q < 0$?

3 Discuss how barium stars are related to S–stars and why they are evidence for wind mass transfer.

Without going into mathematical details, describe how the Bondi–Hoyle–Lyttleton theory of accretion can be used to estimate the rate of mass transfer.

Consider binary systems in which the initial masses of the stars are $5\,M_\odot$ and $0.7\,M_\odot$. Describe, with justification, how the initial period determines whether these evolve to barium stars, cataclysmic variables or Algols.

END OF PAPER