UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 22 June, 2021 12:00 pm to 2:00 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Hyper-accreting black holes

One model for gamma ray bursts involves a disc orbiting around a newly formed black hole and accreting mass at an immense rate. The inner disk is approximated as Keplerian with an inner radius at the ISCO (r_{isco}), the plasma cooled by neutrinos, and its equation of state dominated by radiation and relativistic electrons and positrons:

$$\mathcal{C} = A\rho T^{1/\beta}, \qquad P = \tfrac{11}{12} a T^4.$$

Here $\mathcal{C}$ is the cooling rate per unit volume, and A, β, and a are positive constants. The disk is assumed to be receiving mass at a large outer radius at a constant rate $\dot{M}$.

(a) The diffusion equation governing the evolution of the disc's surface density Σ is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right],$$

in which $\bar{\nu}$ is the mean viscosity.

Give a physical argument for why the disk experiences no torque at $r = r_{\text{isco}}$. If the disc is in steady state, show that $\bar{\nu}\Sigma = \dot{M} f(r)$, where f is a dimensionless function you should find.

(b) At a given radius, the equations of vertical structure include:

$$\frac{\partial P}{\partial z} = -z\Omega^2\rho, \qquad \mathcal{C} = \frac{9}{4}\Omega^2\mu,$$

in which $\mu = \alpha P/\Omega$ is the dynamical viscosity, and α is a constant.

(i) On what timescales do each of these equations establish equilibrium, and how do they compare with the evolutionary timescale of Σ?

(ii) Taking an order of magnitude approach, set $\rho \sim \Sigma/H$, where H is the disk thickness, and solve the first equation for H. By sketching the heating and cooling rates as functions of T, or otherwise, determine that the disk is thermally stable when $0 < \beta < 1/8$.

(iii) Now solve these equations exactly for T, subject to the boundary condition $T = T_0(r)$ at $z = 0$, writing your solution in the form $T = T_0\, g(z/H)^m$, where $g(z/H)$ is a dimensionless function and m is an exponent (both to be determined), and

$$H = \sqrt{\frac{32 A\beta T_0^{1/\beta}}{9\alpha\Omega^3}}.$$

Find similar expressions for ρ and P, and write down relations between their midplane values (ρ_0 and P_0) and T_0.

(c) In the following we set $\beta = 1/2$, and assume that thermal instability is suppressed by an additional process.

(i) Find the exact relationship connecting ρ_0 and Σ and, recalling the definition $\bar{\nu}\Sigma = \int \mu\, dz$, the relationship connecting P_0 and $\bar{\nu}\Sigma$.

(ii) Combining parts (a) and (b), show that $T_0 \propto f^{1/5} r^{-3/4}$, and hence

$$H \propto f^{1/5} r^{3/2}, \quad \text{and} \quad \Sigma \propto f^{3/5} r^{-3/2}.$$

2 Gravitoturbulent protoplanetary discs

Consider a self-gravitating gaseous disc in Keplerian rotation, modelled as razor thin. Small-amplitude, local, and axisymmetric density waves within the disc obey the dispersion relation,

$$\omega^2 = \Omega^2 - 2\pi G\Sigma|k| + c_s^2 k^2,$$

where ω and k are the disturbance's frequency and radial wavenumber, and Ω, Σ, and c_s are the equilibrium disc's orbital frequency, surface density, and sound speed.

(a) Explain the physical origin of each of the three terms on the right side of the dispersion relation.

(b) In the absence of self-gravity, show that the phase speed c_p and group speed c_g of the density waves satisfy $c_p c_g = c_s^2$.

If self-gravity is reinstated, demonstrate that $c_p c_g$ switches sign at a critical wavenumber, which you should find. What are the implications for a localised packet of density waves?

(c) Suppose the disc is gravitationally unstable, with growing modes on the range $k_1 < k < k_2$. Find expressions for k_1 and k_2 in terms of $Q = c_s\Omega/(\pi G\Sigma)$ and $H = c_s/\Omega$. Use these to derive the Toomre instability criterion. For Q much less than 1, show that $k_1 \approx \frac{1}{2}QH^{-1}$ and $k_2 \approx 2Q^{-1}H^{-1}$.

(d) One prescription for the turbulent viscosity $\bar{\nu}$ in a gravitationally unstable disk sets $\bar{\nu} \sim k_1^{-2}\Omega$. Given that the longest unstable mode possesses a wavenumber $\sim k_1$, offer a justification for this prescription.

Show that the prescription suggests $\bar{\nu} \propto r^{9/2}\Sigma^2$, where r is the cylindrical radius to the central star. Next show that the viscous lifetime of the disk is $t_\nu \sim (M_*/M_D)^2\Omega^{-1}$, where M_* and M_D are the initial masses of the star and disk.

(e) Let us now set $\bar{\nu} = Ar^{9/2}\Sigma^2$, where A is a dimensional constant. An initially narrow ring of material will spread under the action of this viscosity, with its outer radius given by $R(t)$ a function of time t. Suppose the inner radius can be taken to 0, and that it experiences no torque.

Show that the constant angular momentum J may be written as

$$J = (M_*G)^{1/2}\, B, \quad \text{where} \qquad B = \int_0^R r^{1/2}\, \Sigma\, 2\pi r dr.$$

Use dimensional analysis to relate R, t, A, and B, and prove that $R \propto t^{2/5}$.

The full similarity solution in this case corresponds to

$$\Sigma = \Sigma_0 \left(\frac{r}{R_0}\right)^{-3/2} \left(\frac{t}{t_0}\right)^{-2/5} \left[1 - \sqrt{\frac{r}{R(t)}}\right]^{1/2},$$

with $R = R_0(t/t_0)^{2/5}$, and R_0, t_0, Σ_0 constants. Show that the total mass in the disc decreases according to

$$M_D = \frac{8}{3}\pi\Sigma_0 R_0^2 \left(\frac{t}{t_0}\right)^{-1/5}.$$

3 Inertial waves and vortex stability

The equations governing fluid motion in the incompressible shearing sheet are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -2\Omega\mathbf{e}_z \times \mathbf{u} - \frac{1}{\rho}\nabla P - \nabla\Phi_t,$$
$$\nabla\cdot\mathbf{u} = 0,$$

where the tidal potential is $\Phi_t = -\frac{3}{2}\Omega^2 x^2$.
(a) Verify that $\mathbf{u} = -\frac{3}{2}\Omega x\mathbf{e}_y$, $P = P_0$ is a solution (P_0 a constant).
Perturb this state with an axisymmetric disturbance $\mathbf{u}'$, P', and derive the four linearised equations governing its evolution:

$$\partial_t u_x' = -\partial_x h' + 2\Omega u_y', \qquad \partial_t u_y' = -\tfrac{1}{2}\Omega u_x',$$
$$\partial_t u_z' = -\partial_z h', \qquad \partial_x u_x' + \partial_z u_z' = 0,$$

where $h' = P'/\rho$. Show that these can be reworked into

$$\partial_t^2(\nabla^2 u_x') + \Omega^2\partial_z^2 u_x' = 0.$$

Assume that the disturbance is $\propto \exp(\mathrm{i}k_x x + \mathrm{i}k_z z - \mathrm{i}\omega t)$, and derive the dispersion relation for inertial waves. When $k_x = 0$ and $k_z \neq 0$, prove that $u_z' = h' = 0$, and then describe the nature of this particular mode.
(b) The core of a Kida vortex in the shearing sheet can be described by

$$\mathbf{u} = \frac{3\Omega}{2(r-1)}\left(\frac{y}{r}\mathbf{e}_x - rx\mathbf{e}_y\right),$$

where r is equal to the ratio of the vortex's semi-major axis to semi-minor axis, with the former aligned with the azimuthal direction.
In the limit $r \to \infty$ verify that the solution converges to circular Keplerian rotation. A fluid blob has position vector $\mathbf{x} = \mathbf{x}(t)$, with $d\mathbf{x}/dt = \mathbf{u}$. Show that the time it takes for the blob to circulate around the vortex is $4\pi(r-1)/(3\Omega)$.
Perturb the vortex solution with a disturbance that only depends on z and t. Write down the linearised horizontal perturbation equations and, on assuming the disturbances are $\propto \exp(-\mathrm{i}\omega t)$, show that

$$\omega^2 = \Omega^2\left[2 - \frac{3}{2r(r-1)}\right]\left[2 - \frac{3r}{2(r-1)}\right].$$

Demonstrate that the mode grows when $3/2 < r < 4$.

END OF PAPER