UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2021 12:00 pm to 3:00 pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider $6 - 8M_\odot$ stars. Decsribe their evolution from the main sequence to the AGB phase including the formation of SAGB (super asymptotic giant branch) stars. Particular attention should be paid to the discussion of the following issues,

(i) the changes in the core composition profile with nuclear burning,

(ii) description of the Schönberg-Chandrasekhar limit,

(iii) dredges-up,

(iv) description of the Härm-Schwarzschild instability and formation of an O/Ne/Mg core for the higher end of the considered masses and

(v) which stars may end up as SAGB stars and what could be their final fate.

2

Consider chemically homogeneous stars. The stellar material is fully ionized ideal gas with mean molecular weight μ. Radiation pressure is negligible. The energy generation rate ϵ is through a variant of the CNO cycle with $\epsilon = \epsilon_0 X \varrho T^{13}$, where ϵ_0 is a constant, ϱ is the density of matter and T is the temperature. The opacity κ is dominated by electron scattering with $\kappa = \kappa_0(1+X)$ where κ_0 is a constant and X is the hydrogen fraction by mass.

(i) Show that for stars of total mass M and radius R, their luminosity L is given by

$$L \propto X \mu^{13} \frac{M^{15}}{R^{16}}$$

(ii) Assume that the stars are fully radiative and show that

$$L \propto \frac{\mu^4}{1+X} M^3$$

(iii) Assume that the stars are fully convective except for a thin, cool, grey atmosphere and show that

$$L \propto \frac{\mu^4}{(1+X)^{8/5}} M^{12/5} R^{6/5}$$

(iv) Assume that the stars discussed in (ii) and (iii) have zero metal content and $X = 1$. Calculate the slope of the lines on the theoretical Hertzsprung-Russell diagram for both radiative and convective stars. For radiative stars of constant mass assume that they are fully mixed as hydrogen burns into helium and find additionally $\mathrm{d}\log L/\mathrm{d}X$ and $\mathrm{d}\log T_{\mathrm{eff}}/\mathrm{d}X$ to determine how these stars evolve from their $X = 1$ line on H-R diagram. Comment on your result.

3

Consider a spherically symmetric star in radiative equilibrium. The stellar material is a perfect gas with gas pressure $P_{\rm g}$ and radiation pressure $P_{\rm rad}$, so the total pressure $P = P(r)$ is $P = P_{\rm g} + P_{\rm rad}$, where r is the radius. The star is chemically homogeneous, its total mass is M, radius R and luminosity L.

Let $\beta = \dfrac{P_{\rm g}}{P}$. Let $\overline{\varrho}(r)$ be the mean density of the material of mass $m_{\rm r}$ within a radius r.

(i) Show that for a configuration in which the mean density $\overline{\varrho}(r)$ interior to r does not increase outward, we have

$$\frac{1}{2}\,G\,(\frac{4}{3}\pi)^{1/3}\,\overline{\varrho}^{4/3}\,m_{\rm r}^{2/3} \leqslant P_{\rm c} - P \leqslant \frac{1}{2}\,G\,(\frac{4}{3}\,\pi)^{1/3}\,\varrho_{\rm c}^{4/3}\,m_{\rm r}^{2/3}\,,$$

where $\varrho_{\rm c}$ is the central density, $P_{\rm c}$ is the central pressure and $\beta_{\rm c}$ is the value of β at the centre.

Derive Eddington's quartic equation.

Show further that for such a configuration $1 - \beta_{\rm c} \leqslant 1 - \beta^*$ where β^* satisfies the equation

$$M = \left(\frac{6}{\pi}\right)^{1/2} \left[\left(\frac{R}{\mu}\right)^4 \frac{3}{a} \left(\frac{1-\beta^*}{\beta^{*4}}\right)\right]^{1/2} G^{-3/2}\,,$$

G is the gravitational constant, a is the radiation constant and μ is the mean molecular weight.

(ii) Assume now that $\nu = \kappa \dfrac{L_{\rm r}}{L}\,\dfrac{M}{m_{\rm r}}$ is a constant, where $L_{\rm r}$ is the luminosity at radius r and κ is the opacity.

Show that in this model β is a constant.

Show that the star is a polytrope and determine the polytropic index.

Derive the Lane-Emden equation and state the boundary conditions.

Use this equation to show that

$$M = \lambda \frac{(1-\beta)^{1/2}}{\beta^2}$$

and determine the constant of proportionality λ for this polytrope.

4

A star is losing mass through a steady, spherically symmetric wind. The wind matter is in radiative equilibrium. The pressure P is given by $P = P_{\rm g} + P_{\rm rad}$, where the gas pressure $P_{\rm g}$ is that of an ideal gas and $P_{\rm rad}$ is the radiation pressure.

Derive a differential equation for the wind velocity $v(r)$ and discuss the topology of its solutions.

Let $\beta = \dfrac{P_{\rm g}}{P}$ and $L_{\rm crit} = \dfrac{4\pi c G m_{\rm r}}{\kappa}$, where $m_{\rm r}$ is the mass contained in a sphere of radius r and κ is the opacity. Assume that the isothermal sound velocity is much smaller than than the free fall velocity.

Use β and $L_{\rm crit}$ to find the conditions necessary for the wind to accelerate to supersonic velocities and show that it is necessary to have $\dfrac{L}{L_{\rm crit}} < 1$.

END OF PAPER