UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 16 June, 2021 12:00 pm to 3:00 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right)=-\rho\boldsymbol{\nabla}\Phi-\boldsymbol{\nabla}p+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}\,,$$

$$\frac{\partial\rho}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\rho=-\rho\boldsymbol{\nabla}\cdot\mathbf{u}\,,$$

$$\frac{\partial p}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}p=-\gamma p\boldsymbol{\nabla}\cdot\mathbf{u}\,,$$

$$\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})\,,$$

$$\nabla^2\Phi=4\pi G\rho\,.$$

In cylindrical polar coordinates (r,ϕ,z),

$$\boldsymbol{\nabla}\Phi=\frac{\partial\Phi}{\partial r}\,\mathbf{e}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\,\mathbf{e}_\phi+\frac{\partial\Phi}{\partial z}\,\mathbf{e}_z\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{1}{r}\frac{\partial}{\partial r}\,(rF_r)+\frac{1}{r}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}\,,$$

$$\boldsymbol{\nabla}\times\mathbf{F}=\left(\frac{1}{r}\frac{\partial F_z}{\partial\phi}-\frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)\mathbf{e}_\phi+\frac{1}{r}\left[\frac{\partial}{\partial r}\,(rF_\phi)-\frac{\partial F_r}{\partial\phi}\right]\mathbf{e}_z\,.$$

In spherical polar coordinates (r,θ,ϕ),

$$\boldsymbol{\nabla}\Phi=\frac{\partial\Phi}{\partial r}\,\mathbf{e}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\,\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\,\mathbf{e}_\phi\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\,(\sin\theta\,F_\theta)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\,,$$

$$\boldsymbol{\nabla}\times\mathbf{F}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\,(\sin\theta\,F_\phi)-\frac{\partial F_\theta}{\partial\phi}\right]\mathbf{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{\partial}{\partial r}\,(rF_\phi)\right]\mathbf{e}_\theta$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\,(rF_\theta)-\frac{\partial F_r}{\partial\theta}\right]\mathbf{e}_\phi\,.$$

1

A star is modelled as a static, self-gravitating, spherically symmetric equilibrium of a perfect gas in the absence of magnetic fields.

Using spherical polar coordinates (r, θ, ϕ), write down the relations that hold between the density $\rho(r)$, the pressure $p(r)$ and the inward radial gravitational acceleration $g(r)$ in the equilibrium state.

Show that small perturbations from this basic state satisfy the linearized equation of motion

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\boldsymbol{\nabla}\delta\Phi - \delta\rho\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\delta p\,,$$

where $\boldsymbol{\xi}$ is the displacement and the Eulerian perturbations are given by

$$\begin{aligned}\delta\rho &= -\rho\Delta - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho\,,\\ \delta p &= -\gamma p\Delta - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}p\,,\\ \nabla^2\delta\Phi &= 4\pi G\,\delta\rho\,,\end{aligned}$$

where $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$.

Let $Y_l^m(\theta,\phi)$ be a spherical harmonic function such that

$$\nabla^2 Y_l^m = -\frac{l(l+1)}{r^2}\,Y_l^m\,,$$

where l and m are integers with $l \geqslant |m|$. If the displacement has the form

$$\boldsymbol{\xi} = \mathrm{Re}\left\{\left[\tilde{\xi}_r(r)\,Y_l^m\,\mathbf{e}_r + \tilde{\xi}_\mathrm{h}(r)\,\boldsymbol{\nabla}Y_l^m\right]e^{-i\omega t}\right\}\,,$$

show that

$$\Delta = \mathrm{Re}\left[\tilde{\Delta}(r)\,Y_l^m\,e^{-i\omega t}\right] \qquad \text{with} \qquad \tilde{\Delta} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\tilde{\xi}_r\right) - k_\mathrm{h}^2\tilde{\xi}_\mathrm{h}\,,$$

where the horizontal wavenumber $k_\mathrm{h}(r)$ is defined by $k_\mathrm{h}^2 = l(l+1)/r^2$. By writing

$$\delta\rho = \mathrm{Re}\left[\widetilde{\delta\rho}(r)\,Y_l^m\,e^{-i\omega t}\right]\,,$$

and similarly for the other scalar perturbations, deduce that the linearized equations reduce to the ordinary differential equations

$$\begin{aligned}-\rho\omega^2\tilde{\xi}_r &= -\rho\frac{d\,\widetilde{\delta\Phi}}{dr} - g\,\widetilde{\delta\rho} - \frac{d\,\widetilde{\delta p}}{dr}\,,\\ -\rho\omega^2\tilde{\xi}_\mathrm{h} &= -\rho\,\widetilde{\delta\Phi} - \widetilde{\delta p}\,,\\ \widetilde{\delta\rho} &= -\rho\tilde{\Delta} - \tilde{\xi}_r\frac{d\rho}{dr}\,,\\ \widetilde{\delta p} &= -\gamma p\tilde{\Delta} - \tilde{\xi}_r\frac{dp}{dr}\,,\\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\,\widetilde{\delta\Phi}}{dr}\right) - k_\mathrm{h}^2\,\widetilde{\delta\Phi} &= 4\pi G\,\widetilde{\delta\rho}\,.\end{aligned}$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Show that the radial equation of motion can be rewritten as

$$(\omega^2 - N^2)\tilde{\xi}_r = \frac{d}{dr}\left(\widetilde{\delta\Phi} + \frac{\widetilde{\delta p}}{\rho}\right) - \frac{N^2}{g}\frac{\widetilde{\delta p}}{\rho}\,,$$

where $N^2(r)$ is a quantity that you should define. Explain why, near the centre of a star like the Sun,

$$N^2 \approx Ar^2\,,$$

where A is a positive constant.

2

(a) Consider the one-dimensional flow of a perfect gas (of adiabatic exponent γ), with velocity $\mathbf{u} = u_x(x,t)\,\mathbf{e}_x$, in the absence of gravity and magnetic fields. Write down the conservative forms of the partial differential equations for mass, momentum and total energy. What typical values of γ are expected in astrophysical fluids?

(b) Deduce the Rankine–Hugoniot relations that connect the fluid variables on either side of a stationary, normal shock. Show that their solution is

$$\frac{u_{x1}}{u_{x2}} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}_{x1}^2}{(\gamma-1)\mathcal{M}_{x1}^2+2}\,, \qquad \frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_{x1}^2-(\gamma-1)}{(\gamma+1)}\,,$$

where $\mathcal{M}_x = u_x/v_\mathrm{s}$ is the Mach number of the flow normal to the shock, and the subscripts 1 and 2 refer to the regions upstream and downstream of the shock, respectively.

(c) By making a Galilean transformation, or otherwise, show that when an oblique shock is considered by including a tangential velocity u_y in the analysis, the above relations still hold, along with $u_{y1} = u_{y2}$.

(d) A weak shock can be defined as one in which

$$\frac{p_2}{p_1} = 1+\epsilon, \qquad \text{with} \quad 0 < \epsilon \ll 1\,.$$

Consider a weak oblique shock and let $\mathcal{M}_1 = |\mathbf{u}_1|/v_{\mathrm{s}1} > 1$ be the Mach number of the *total* upstream flow. Show that, to a first approximation, the angle β between the velocity vector and the shock front is given by

$$\beta_1 \approx \beta_2 \approx \arcsin\left(\mathcal{M}_1^{-1}\right)\,.$$

By showing that

$$\frac{\tan\beta_1}{\tan\beta_2} \approx 1 + \frac{\epsilon}{\gamma}$$

to first order in ϵ, or otherwise, show that the flow is deflected by a small angle

$$\theta \approx \frac{\epsilon}{\gamma}\frac{\sqrt{\mathcal{M}_1^2-1}}{\mathcal{M}_1^2}$$

as a result of passing through the shock front. Is the flow turned towards or away from the shock front?

3

(a) Let (r, ϕ, z) be cylindrical polar coordinates and consider the vector field

$$\mathbf{A} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\phi + \beta\boldsymbol{\nabla}\phi\,, \qquad (*)$$

where $\alpha(r, z)$ and $\beta(r, z)$ are two axisymmetric scalar fields. [Note that $\boldsymbol{\nabla}\phi = \mathbf{e}_\phi/r$.]

By working explicitly in cylindrical polar coordinates, show that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ and

$$\boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla}\phi + (\mathcal{L}\alpha)\boldsymbol{\nabla}\phi\,,$$

where the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L}\alpha = -r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\alpha}{\partial r}\right) - \frac{\partial^2\alpha}{\partial z^2}\,.$$

(b) Suppose that the magnetic vector potential has the form $(*)$. Using the results of part (a), write down expressions for the magnetic field $\mathbf{B}$ and the electric current density $\mathbf{J}$ in terms of α and β. Deduce that the Lorentz force per unit volume, $\mathbf{F}_\mathrm{m}$, is given by

$$r^2\mu_0\mathbf{F}_\mathrm{m} = (\mathcal{L}\beta)\boldsymbol{\nabla}\beta - (\mathcal{L}\alpha)\boldsymbol{\nabla}(\mathcal{L}\alpha) + f\,\mathbf{e}_\phi\,,$$

where

$$f = [\boldsymbol{\nabla}(\mathcal{L}\alpha) \times \boldsymbol{\nabla}\beta] \cdot \mathbf{e}_\phi\,.$$

[You may assume the vector identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.]

(c) For an axisymmetric magnetostatic equilibrium state, explain why f must vanish. Deduce that this implies that

$$\mathcal{L}\alpha = F(\beta)\,,$$

where $F(\beta)$ is an arbitrary function.

(d) Suppose that the equilibrium is barotropic, such that $\boldsymbol{\nabla}p = \rho\boldsymbol{\nabla}h$, where p and ρ are the pressure and density and $h(r, z)$ is some function. Show further that

$$\mathcal{L}\beta = F\frac{dF}{d\beta} + r^2\rho\,G(\beta)\,,$$

where $G(\beta)$ is an arbitrary function.

4

A magnetized outflow from a rotating astrophysical body is modelled as a steady, axisymmetric flow in ideal MHD, using cylindrical polar coordinates (r, ϕ, z).

(a) By writing the magnetic field in the form

$$\mathbf{B} = \mathbf{B}_\mathrm{p} + B_\phi \, \mathbf{e}_\phi \, ,$$

where

$$\mathbf{B}_\mathrm{p} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi = -\frac{1}{r}\mathbf{e}_\phi \times \boldsymbol{\nabla}\psi$$

is the poloidal magnetic field and $\psi(r, z)$ is the magnetic flux function, show that the MHD equations imply

$$\begin{aligned} \rho\mathbf{u}_\mathrm{p} &= k\mathbf{B}_\mathrm{p} \, , \\ \frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} &= \omega \, , \\ ru_\phi - \frac{rB_\phi}{\mu_0 k} &= \ell \, , \\ \frac{1}{2}|\mathbf{u}|^2 + \Phi + h - \frac{r\omega B_\phi}{\mu_0 k} &= \varepsilon \, , \end{aligned}$$

where k, ω, ℓ, ε and the specific entropy s are constant along each magnetic field line, the subscript p denotes the poloidal part and h is the specific enthalpy.

[You may quote the conservative form of the total energy equation in MHD.]

(b) Solve the simultaneous equations for u_ϕ and B_ϕ in the case of an outflow that passes smoothly through an Alfvén point.

(c) Consider an open field line on which $r^2|\mathbf{B}_\mathrm{p}|$ tends to a non-zero limit F as $r \to \infty$. Assuming that $|\mathbf{u}_\mathrm{p}|$ tends to a non-zero terminal speed u_∞ as $r \to \infty$, and making plausible assumptions about h and Φ where necessary, show that the azimuthal Alfvén speed $v_{\mathrm{a}\phi}$ tends to a constant as $r \to \infty$, and that the terminal speed satisfies the equation

$$\varepsilon = \frac{1}{2}u_\infty^2 + \frac{\omega^2 F}{\mu_0 k}\frac{1}{u_\infty} \, .$$

END OF PAPER