UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 22 June, 2021 12:00 pm to 2:00 pm

PAPER 313

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

*Attempt no more than **TWO** questions.*
*There are **THREE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Show, by constructing an atlas consisting of two charts, and using a stereographic projection, that a sphere

$$S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}$$

is a smooth manifold of dimension n.

Prove that any Lie group is a parallelizable manifold.

Construct the general form of an element of the the group $SU(2)$, and deduce that S^3 is parallelizable.

Show that there exists at least one more $n \neq 3$ such that S^n is parallelizable.

2

Define the Lie group $SO(3)$, and find a basis for its Lie algebra.

Show that the action $\mathbf{r} \to A\mathbf{r}$ of $SO(3)$ on $\mathbb{R}^3$ is generated by three vector fields

$$V_a = \sum_{b,c=1}^{3} \epsilon_{abc} x^b \frac{\partial}{\partial x^c}, \quad a = 1, 2, 3, \tag{1}$$

and show that these vector fields span a Lie algebra.

Show that the vector fields (1) are Hamiltonian with respect to the Poisson structure on $\mathbb{R}^3$ given by

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y$$

and find the corresponding Hamiltonians.

Find a non–constant function $F : \mathbb{R}^3 \to \mathbb{R}$ such that $\{F, x^a\} = 0, a = 1, 2, 3$. Hence, or otherwise show that the group action descends to a symplectic action on a two–sphere $S^2 \subset \mathbb{R}^3$.

3

Let A be an $\mathfrak{su}(2)$ valued one–form on $\mathbb{R}^D$. Define the 2nd Chern form C_2, and show that $C_2 = dY$, where

$$Y = \frac{1}{8\pi^2}\mathrm{Tr}(A \wedge dA + cA \wedge A \wedge A)$$

where c is a constant which should be determined.

Show that if $g : \mathbb{R}^D \to SU(2)$ and

$$A \to gAg^{-1} - dg \cdot g^{-1} \tag{1}$$

then

$$F = dA + A \wedge A \to gFg^{-1}, \quad \text{and} \quad C_2 \to C_2.$$

State the boundary conditions satisfied by an $SU(2)$ Yang–Mills instanton on $\mathbb{R}^4$, and express the instanton number in terms of an integral involving the gauge transformation at infinity.

END OF PAPER