UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 16 June, 2021 12:00 pm to 3:00 pm

PAPER 312

FIELD THEORY IN COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For a massless scalar field φ in de Sitter spacetime with $c_s = 1$, consider the interaction

$$S \supset \int dt d^3x \, a^3 \frac{\lambda}{3!} \dot{\varphi}^2 \varphi \,,$$

where a dot denotes a derivative with respect to cosmological time t. Do you expect that this interaction can induce large non-Gaussianity in single-field slow-roll inflation? Justify your answer.

(b) Rewrite this interaction in terms of conformal time τ and compute the bispectrum B of φ at $\tau = 0$, using the de Sitter mode functions

$$f_k(\tau) = \frac{H}{\sqrt{2k^3}}(1 + ik\tau)e^{-ik\tau} \,.$$

(c) Starting from the scale invariant property

$$\langle \phi(\mu\mathbf{x}_1)\phi(\mu\mathbf{x}_2)\phi(\mu\mathbf{x}_3)\rangle = \langle \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\rangle \,,$$

where μ is a positive real number, derive how the bispectrum $B(k_1, k_2, k_3)$ scales as $k_a \to \mu k_a$ for $a = 1, 2, 3$. Show that the bispectrum you computed above obeys this scaling.

2

Consider a scalar field ϕ whose action is invariant under a shift symmetry,

$$\Delta\phi(\mathbf{x}) = c\,,$$

for any constant c. On a background $\langle\phi\rangle = 0$, compute the corresponding soft theorem for the bispectrum of ϕ as follows:

(a) Derive the transformation of $\phi(\mathbf{k})$. The generator Q of the above symmetry obeys

$$i[Q,\phi] = \Delta\phi\,. \qquad (1)$$

Write down Q in terms of the momentum conjugate Π of ϕ.

(b) Using the free field expressions for Π and ϕ as well as the Wronskian condition derive that

$$i\langle[Q,\phi(\mathbf{k})\phi(\mathbf{k}')]\rangle = \frac{1}{P(0)}\langle\phi(\mathbf{k})\phi(\mathbf{k}')\phi(\mathbf{0})\rangle\,,$$

where $\phi(\mathbf{0}) = \phi(\mathbf{k}=\mathbf{0})$ and $P(k)$ is the power spectrum of ϕ.

(c) Show that the correlator $\langle\Delta(\phi(\mathbf{k})\phi(\mathbf{k}'))\rangle$ vanishes. Hence, using (1), determine the soft theorem for the bispectrum of ϕ.

(d) State the soft theorem obeyed by bispectrum of the curvature perturbations $\mathcal{R}$ in single-field inflation. Compare that soft theorem with the one you derived above.

3

(a) In Newtonian perturbation theory for Dark Matter, consider the continuity and Euler equations

$$\delta' + \nabla\left[(1+\delta)\,\boldsymbol{v}\right] = 0\,,$$
$$v_i' + \mathcal{H}v_i + \boldsymbol{v}\cdot\nabla v_i = -\nabla_i\phi - \frac{1}{\rho}\nabla_j\left(\rho\sigma_{ij}\right)\,.$$

Briefly discuss their physical meaning and define each of the variables. Assuming a curl-free velocity and a vanishing velocity dispersion, derive the coupling kernels α and β.

(b) Draw all the diagrams to compute the matter power spectrum up to fourth order (one loop) and write down the corresponding expressions in terms of the convolution kernels F_n. Draw a plot of the various contributions to the matter power spectrum today at this order, carefully labelling the axis and specifying some reference points on each axis.

(c) In the Effective Field Theory of Large Scale Structures, write down the counterterms needed to renormalize the one-loop power spectrum. Using that

$$\lim_{k\ll q} F_3(\mathbf{k},\mathbf{q},-\mathbf{q}) \propto \frac{k^2}{q^2}\,,$$

show that the counterterm related to the speed of sound has the correct momentum scaling to cancel the cutoff dependence of one of the one-loop contributions to the power spectrum.

(d) Consider a local Lagrangian bias model

$$\delta_g^{(L)}(\mathbf{q}) = b_1^{(L)}\delta(\mathbf{q})\,,$$

where $\delta(\mathbf{q})$ represents the Dark Matter inhomogeneities in Lagrangian space and is assumed to be Gaussian. Derive the linear Eulerian bias $b_1^{(E)}$ in terms of $b_1^{(L)}$.

4

Compute the left-hand side of the Boltzmann equations for the photons of the Cosmic Microwave Background (CMB) to first order in perturbations as follows.

(a) Consider the metric in Newtonian gauge,

$$ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2(1-2\Phi)\delta_{ij}dx^i dx^j \,,$$

and work to linear order in the Newtonian potentials Ψ and Φ. In the absence of anisotropic stress, how are Ψ and Φ related? You may assume this relation in the following. Derive the following expression of the photon momentum

$$P^\mu = \frac{\epsilon}{a^2}\left[1-\Psi, (1+\Phi)\hat{p}^i\right] \,,$$

where $p^2 \equiv g_{ij}P^iP^j$, $\epsilon \equiv pa$ and $\hat{p}^i$ is a unit-norm vector ($\hat{p}^i\delta_{ij}\hat{p}^j = 1$) pointing in the P^i direction. Using the following expressions for the Christoffel symbols,

$$\Gamma^0_{00} = \mathcal{H} + \Psi' \qquad \Gamma^0_{0i} = \partial_i\Psi \qquad \Gamma^0_{ij} = \left[\mathcal{H} - \Phi' - 2\mathcal{H}(\Psi+\Phi)\right]\delta_{ij} \,,$$

compute $d\ln\epsilon/d\eta$ to first order in perturbations.

(b) Let f be the photon distribution function

$$f = \left\{\exp\left[\frac{\epsilon}{a\bar{T}(1+\Theta)}\right] - 1\right\}^{-1} .$$

where $\Theta = \Theta(\eta, \mathbf{x}, \hat{\mathbf{p}})$ are temperature perturbations. Re-write $df/d\eta$ in terms of partial derivatives. Expand each term to first order in perturbations Φ, Ψ and Θ.

(c) Draw a plot of the total power spectrum of CMB temperature anisotropies as well as the Sachs-Wolfe and Doppler contributions. Carefully label the axis and indicate on them a couple of reference points with the appropriate units. Briefly discuss what *diffusion damping* is and how it is visible in this plot.

END OF PAPER