UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 10 June, 2021 12:00 pm to 3:00 pm

PAPER 311

BLACK HOLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) (i) What is a *null geodesic congruence*? Let U^a be tangent to the affinely parameterized geodesics of a null geodesic congruence. Let $B_{ab} = \nabla_b U_a$. Explain why $U^a B_{ab} = B_{ab} U^b = 0$. [2]

(ii) Explain how to construct a vector field N^a such that $N^2 = 0$, $U \cdot N = -1$ and $U \cdot \nabla N^a = 0$. [2]

(iii) Let $P^a{}_b = \delta^a{}_b + U^a N_b + N^a U_b$ and $\widehat{B}^a{}_b = P^a{}_c B^c{}_d P^d{}_b$. Explain how to define the *expansion* θ, *rotation* $\widehat{\omega}_{ab}$ and *shear* $\widehat{\sigma}_{ab}$ of the congruence in terms of $\widehat{B}^a{}_b$. [1]

(iv) Consider a null geodesic congruence containing the generators of a null hypersurface $\mathcal{N}$. Show that the rotation vanishes on $\mathcal{N}$. *[You may assume Frobenius theorem.]* [5]

(b) Derive Raychaudhuri's equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \widehat{\sigma}^{ab}\widehat{\sigma}_{ab} + \widehat{\omega}^{ab}\widehat{\omega}_{ab} - U^a U^b R_{ab}\,,$$

for an affinely parametrised null congruence of geodesics with tangent vector U^a. [7]

(c) Assume the spacetime satisfies the Einstein equation with matter obeying the null energy condition. Show that if $\theta = \theta_0 < 0$ at a point p on a generator γ of a null hypersurface, then $\theta \to -\infty$ within finite affine parameter distance $2/|\theta_0|$, provided γ extends this far. [3]

(d) Show that if T is a trapped surface in a strongly asymptotically predictable spacetime obeying the null energy condition, then T must be contained inside a black hole region. *[You may assume that strong asymptotic predictability can be used to show that $\dot{J}^+(T)$ intersects $\mathcal{I}^+$ and that every $p \in \dot{J}^+(T)$ lies on a future-directed null geodesic starting from T, orthogonal to T, with no point conjugate to T between T and p. You may also assume that if $\theta \to -\infty$ at a point q on a null geodesic γ through p, then q is conjugate to p along γ.]* [10]

2 Consider the following line element for a six-dimensional rotating black hole solution in Boyer-Lindquist type coordinates $(t, r, \theta, \phi, \lambda, \psi)$

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{\Sigma(r,\theta)}\left(\mathrm{d}t - a\,\sin^2\theta\,\mathrm{d}\phi\right)^2 + \Sigma(r,\theta)\left[\frac{\mathrm{d}r^2}{\Delta(r)} + \mathrm{d}\theta^2\right]$$
$$+\frac{\sin^2\theta}{\Sigma(r,\theta)}\left[a\,\mathrm{d}t - \left(r^2+a^2\right)\mathrm{d}\phi\right]^2 + r^2\cos^2\theta\left(\mathrm{d}\lambda^2 + \sin^2\lambda\,\mathrm{d}\psi^2\right)\,,$$

with

$$\Delta(r) = r^2 + a^2 - \frac{r_+}{r}(r_+^2 + a^2)\,, \qquad \Sigma(r,\theta) = r^2 + a^2\cos^2\theta\,.$$

and $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \lambda \leqslant \pi$, $\psi \sim \psi + 2\pi$ and $\phi \sim \phi + 2\pi$.

You may assume that

$$\sqrt{-g} = r^2\,\sin\theta\,\cos^2\theta\,\sin\lambda\,\Sigma(r,\theta)$$

and

$$R^{abcd}R_{abcd} = \frac{12r_+^2\left(a^2+r_+^2\right)^2}{r^6\Sigma(r,\theta)^6}\left[\Sigma(r,\theta)^4 + r^2\Sigma(r,\theta)^3 + 2r^4\Sigma(r,\theta)^2 - 16r^6\Sigma(r,\theta) + 32r^8\right]\,.$$

(a) Show that

$$K = \frac{\partial}{\partial t} \quad \text{and} \quad m = \frac{\partial}{\partial\phi}$$

are Killing vectors. [2]

(b) Compute the Komar mass M of this solution defined by

$$M = -\frac{1}{12\pi}\lim_{r\to+\infty}\int \star\,\mathrm{d}K\,,$$

where the integral is taken over a constant t, r surface and the orientation is $\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\lambda\wedge\mathrm{d}\psi$. [10]

(c) By introducing Kerr like coordinates $(v, r, \theta, \chi, \lambda, \psi)$ with

$$\mathrm{d}v = \mathrm{d}t + \frac{r^2+a^2}{\Delta(r)}\mathrm{d}r \quad \text{and} \quad \mathrm{d}\chi = \mathrm{d}\phi + \frac{a}{\Delta(r)}\mathrm{d}r$$

show that $r = r_+$ is a null hypersurface with normal $\xi = K + \Omega_H\,m$ where $\Omega_H = a/(r_+^2 + a^2)$. [8]

(d) Compute the surface gravity associated with ξ, and the area A of the intersection of the black hole event horizon with a partial Cauchy surface of constant v. [6]

(e) Sketch the Penrose diagram for the spacetime described by this metric on the submanifold $\theta = \lambda = 0$. [4]

3

(a) State and prove the version of the first law of black hole mechanics that relates the change in area of the event horizon to the energy and angular momentum of infalling matter. *[You may assume Raychaudhuri's equation, knowledge of Gaussian null coordinates and the zeroth law of black hole mechanics.]* [10]

(b) State the second law of black hole mechanics and sketch a proof. *[You may assume that the generators of the future horizon are complete to the future and standard results on conjugate points.]* [5]

(c) Consider two identical and initially far apart Kerr black holes with parameters (M, J), which merge to form a Schwarzschild black hole with mass M'. The fraction of the initial energy radiated in gravitational waves is

$$\eta = \frac{2M - M'}{2M}\,.$$

Use the second law to derive an upper bound on η in terms of J/M^2. What is the initial configuration that achieves the highest η? *[You may assume that the area of the intersection of the future event horizon of a Kerr black hole, with mass M and angular momentum J, and a partial Cauchy surface is given by $A = 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)$.]* [5]

(d) Let $(\mathcal{M}, g)$ be a strongly asymptotically predictable spacetime and $(\overline{\mathcal{M}}, \overline{g})$ be its conformal compactification. Let Σ_1 and Σ_2 be Cauchy surfaces for an open region $\overline{V} \subset \overline{\mathcal{M}}$ with $\Sigma_2 \subset I^+(\Sigma_1)$ and $\overline{\mathcal{M} \cap J^-(\mathcal{I}^+)} \subset \overline{V}$. Let $\mathcal{B}$ be the black hole region and B be a connected component of $\mathcal{B} \cap \Sigma_1$. Show that $J^+(B) \cap \Sigma_2$ is contained within a connected component of $\mathcal{B} \cap \Sigma_2$. Explain in which sense this theorem ensures that a black hole cannot split. [10]

4

(a) Describe how a free massive real scalar field is quantised in a stationary and globally hyperbolic spacetime $(\mathcal{M}, g)$. Define the *vacuum state.* [6]

(b) Suppose a spacetime $(\mathcal{M}, g)$ has the form of a sandwich: there are two non-intersecting Cauchy surfaces, Σ_1 and Σ_2 such that the spacetime is stationary to the past of Σ_1 and to the future of Σ_2, and in between it is time dependent.

(i) Describe how this may lead to particle production. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by an observer in the far future, in the vacuum state as defined by an observer in the far past. [6]

(ii) Why is this sandwich spacetime example relevant to the derivation of Hawking radiation in the spacetime of gravitational collapse to a black hole? [2]

(c) Consider the Kerr black hole in Kerr coordinates (v, r, θ, χ)

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{\Sigma(r,\theta)}\left(\mathrm{d}v - a\,\sin^2\theta\,\mathrm{d}\chi\right)^2 + 2\mathrm{d}v\mathrm{d}r - 2\,a\,\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\chi + \Sigma(r,\theta)\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma(r,\theta)}\left[a\,\mathrm{d}v - \left(r^2+a^2\right)\mathrm{d}\chi\right]^2 ,$$

with

$$\Delta(r) = (r-r_+)(r-r_-) \quad \text{and} \quad \Sigma(r,\theta) = r^2 + a^2\cos^2\theta$$

where $r_\pm = M \pm \sqrt{M^2-a^2}$, with $|a| \leqslant M$ and $a = J/M$.

(i) Assuming that $r = r_+$ is a Killing horizon of

$$\xi = \frac{\partial}{\partial v} + \Omega_H\,\frac{\partial}{\partial \chi}$$

determine Ω_H and the Hawking temperature T_H of the Kerr black hole. [4]

(ii) Consider such a black hole in thermal equilibrium with an infinite reservoir of radiation at temperature T_H and take $a = 0$. Explain why the black hole is thermodynamically unstable. [2]

(iii) The specific heat of a rotating black hole of mass M and fixed angular momentum J is

$$C_J = T_H\,\left.\frac{\partial S_{\mathrm{BH}}}{\partial T_H}\right|_J .$$

Using the Bekenstein-Hawking entropy relation determine C_J. *[Hint: it might be useful to express the spin parameter a as $a^2 = r_+r_-$.]* [8]

(iv) Find the range of values of $|a|$ for fixed M for which the black hole is in stable equilibrium with an infinite reservoir of radiation at its Hawking temperature. [2]

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