UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2021 12:00 pm to 3:00 pm

PAPER 310

COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question you will compare the predictions of an Einstein de Sitter universe (which has a matter density parameter $\Omega_{m,0} = 1$ today) and a simplified LCDM cosmology that you can assume is described fully by density parameters today of $\Omega_{\Lambda,0} = 0.7$ and $\Omega_{m,0} = 0.3$. You may assume throughout the question that radiation density and curvature are negligible. You may also assume the Friedmann and continuity equations $(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3}$, $(\frac{\ddot{a}}{a}) = -\frac{4\pi G}{3}(\rho + 3P)$, and $d\rho = -3\frac{da}{a}(\rho + P)$.

(a) From the continuity equation, demonstrate that for a component with equation of state parameter w, the energy density ρ evolves with scale factor a as

$$\rho \propto a^{-3(1+w)} \tag{1}$$

Hence show that the Hubble parameter at redshift z can be written as

$$H(z) = H_0 \left[\sum_i \Omega_{i,0}(1+z)^{3(1+w_i)} \right]^{1/2} \tag{2}$$

where you should define the density parameter $\Omega_{i,0}$, the sum is over components i with equation of state parameters w_i, and H_0 is the Hubble constant.

(b) Show that the deceleration parameter q, which is defined as $q \equiv -\frac{\ddot{a}a}{(\dot{a})^2}$, can be written as

$$q = \frac{1}{2}\sum_i \Omega_i(z)(1+3w_i), \tag{3}$$

where $\Omega_i(z)$ is the density parameter of component i at redshift z.

From supernova lightcurves we have determined that the deceleration parameter has a negative value today. Is this consistent with an Einstein de Sitter cosmology or with the simplified LCDM cosmology described above?

(c) Derive an expression for the age of the universe in terms of the matter density parameter today $\Omega_{m,0}$, the dark energy density parameter today $\Omega_{\Lambda,0}$, and the Hubble constant H_0 (you may leave your result as an integral). Deduce that for an Einstein de Sitter universe the age of the universe t_0 is

$$t_0 = \frac{2}{3}H_0^{-1} \tag{4}$$

(d) The ages of the oldest stars in globular clusters have been measured to be 12 billion years; the Hubble constant has also been measured to have a value such that $H_0^{-1} = 14$ billion years. Are these measurements compatible within an Einstein de Sitter cosmology? Justify your answer. Briefly and qualitatively discuss whether a LCDM cosmology could provide better consistency between these measurements.

2 In this question you will discuss inflation and its end. You may assume that the energy momentum tensor for a scalar inflaton field ϕ with potential $V(\phi)$ is $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)$ and that for a comoving observer $T_{00} = \rho_\phi$ and $T_{ij} = -P_\phi g_{ij}$ (using a $+ \; - \; - \; -$ metric signature). You may further neglect fluctuations in ϕ and assume that its value is independent of spatial position.

(a) From the energy momentum tensor, show that the energy density ρ_ϕ and pressure P_ϕ of the inflaton field are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

(b) Using the Friedmann equations or otherwise, deduce the Klein-Gordon equation for the inflaton:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \tag{3}$$

Write down approximate equations for the evolution of ϕ and of H during slow-roll inflation.

[Hint: the second Friedmann equation is given by $\dot{H} + H^2 = (\frac{\ddot{a}}{a}) = -\frac{4\pi G}{3}(\rho + 3P)$.]

(c) Now consider an inflation model with a potential $V(\phi) = V_0\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{pl}}}\right)^2$. By considering the potential slow roll parameters, show that for $\phi \ll \sqrt{\frac{3M_{pl}^2}{2}}$ the potential does not allow slow-roll inflation but that for $\phi \gg \sqrt{\frac{3M_{pl}^2}{2}}$ it does admit slow-roll inflation.

[Hint: the potential slow-roll parameters are $\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2$ and $\eta_V = M_{pl}^2\left(\frac{V_{,\phi\phi}}{V}\right)$]

(d) After inflation has ended in this model, the $3H\dot{\phi}$ term in the Klein-Gordon equation can be neglected. By expanding the potential to leading order in ϕ around its minimum, write down an approximate oscillatory equation of motion for ϕ after inflation has ended. Evaluate an average of the inflaton field's energy density and pressure over many oscillations and hence show that the average energy density is expected to fall as $\rho_\phi \propto a^{-3}$ after inflation has ended.

(e) Are there any potentials $V(\phi)$ for which you would not expect this $\rho_\phi \propto a^{-3}$ evolution after the end of inflation? Justify your answer.

3

(a) Explain in detail why the temperature of the cosmic neutrino background today T_ν is related to the CMB photon temperature T today by

$$\frac{T_\nu}{T} = \left(\frac{4}{11}\right)^{1/3}. \tag{1}$$

In your explanation, you may assume without proof that the entropy $S = \frac{\rho+P}{T}V$ is conserved in an expanding universe.

(b) After neutrino decoupling, the relic neutrinos retain the relativistic Fermi-Dirac distribution function (even if the individual neutrinos become non-relativistic), which is given by

$$f(p) = \frac{1}{e^{ap/T_\nu} + 1} \tag{2}$$

in terms of the magnitude of the momentum p.

Write down an integral expression for the energy density of one relic neutrino species and show that in the limit of early times (small scale factor a) where $T_\nu/a \gg m_\nu$, with m_ν the small mass of this neutrino species, the energy density is:

$$\rho_\nu = \frac{7\pi^2}{120}(T_\nu)^4/a^4. \tag{3}$$

[Hint: you may assume that $g = 2$ for this neutrino species and that the density of particles in phase space is $\frac{g}{(2\pi)^3}f(p)$. You may also use the standard integrals: $\int_0^\infty \frac{x^a}{e^x+1} = a!\left(1 - \frac{1}{2^a}\right)\zeta(a+1)$, where $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) \approx 1.202$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(5) \approx 1.037$]

(c) Similarly show, by expanding to leading order in $T_\nu/(am)$, that at late times where $T_\nu/a \ll m_\nu$, the energy density becomes

$$\rho_\nu \approx n_\nu m_\nu \left(1 + F \times (T_\nu/(am_\nu))^2\right), \tag{4}$$

where F is a constant you should specify and where the neutrino species' number density is $n_\nu = \frac{3\zeta(3)}{2\pi^2}(T_\nu/a)^3$.

(d) Use the result in (c) to calculate an order of magnitude estimate (written in terms of T_ν, a and m_ν) for the mean velocity of the neutrinos at late times where $T_\nu/a \ll m_\nu$.

(e) A far-future experiment is able to measure the neutrino flux from the cosmic neutrino background as a function of position on the sky. Explain why, in principle, this contains information about density fluctuations from times earlier than CMB recombination. In analogy with the CMB, we can define a cosmic neutrino background last scattering surface that describes where and when these neutrinos decoupled; would you always expect the neutrino last scattering surface to lie at a larger comoving distance from Earth than the CMB last-scattering surface? Justify your answer.

4 In this question you will discuss perturbations on superhorizon scales.

(a) Consider a standard single-field slow-roll inflation model, where ϕ is the inflation field and $V(\phi)$ is its potential. You may assume that $a(\tau) = -(H\tau)^{-1}$ (with τ the conformal time) and that $H = \sqrt{\frac{V(\phi)}{3M_{\rm pl}^2}} \approx$ constant. Canonical quantization leads to the following expressions for the field operator $\hat{f} = a\hat{\delta\phi}$ and its conjugate momentum $\hat{\pi}$, describing perturbations to the inflation field $\delta\phi$:

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[f_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + f^*_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right]$$

$$\hat{\pi}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[f'_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + (f^*)'_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right]$$

where $f^*_{\mathbf{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau})$ and $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}$ are lowering and raising operators. State the commutation relations obeyed by $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}'}$. By calculating the two point correlation function of $\delta\phi$, show that the dimensionless power spectrum of $\delta\phi$ after horizon exit is

$$\Delta^2_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \tag{1}$$

[Hint: you may assume that the dimensionless power spectrum $\Delta^2_{\delta\phi}$ is related to the two point correlation function via $\langle 0|\hat{\delta\phi}(\tau, \mathbf{x})\hat{\delta\phi}(\tau, \mathbf{x} + \mathbf{r})|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{2\pi^2}{k^3} \Delta^2_{\delta\phi} e^{-i\mathbf{k}\cdot\mathbf{r}}$.]

(b) Assuming the model from (a), calculate the commutator of $\hat{f}$ and $\hat{\pi}$ on superhorizon scales. Given your results, are perturbations well described by classical physics on superhorizon scales?

(c) Assuming the model from (a), evaluate $\hat{f}$ and $\hat{\pi}$ on superhorizon scales to leading order in $k\tau$. Using your results, approximately evaluate the commutator of $\hat{f}$ and $\hat{\pi}$ on superhorizon scales and comment on the significance of your answer.

(d) Show that when all perturbations are adiabatic, the curvature perturbation $\mathcal{R}$ is conserved outside the horizon. Now consider also the presence of an entropy perturbation $\mathcal{S}$ in the universe consisting only of CDM and photons, defined by:

$$\mathcal{S} \equiv \delta_c - \frac{3}{4}\delta_\gamma. \tag{2}$$

Show that when $\mathcal{S}$ is non-zero, the curvature perturbation on superhorizon scales evolves as:

$$\frac{d\mathcal{R}}{d\ln a} = -f(\bar{\rho}_\gamma, \bar{\rho}_c)\mathcal{S}, \tag{3}$$

where you should specify the function $f(\bar{\rho}_\gamma, \bar{\rho}_c)$.
[Hint: you may assume that on superhorizon scales $(\bar{\rho} + \bar{P})\frac{\mathcal{R}'}{\mathcal{H}} = -(\delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho)$, where ρ and P are the total energy density and pressure.]

END OF PAPER