UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2021 12:00 pm to 3:00 pm

PAPER 309

GENERAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)(i) Let X be a vector field and T a tensor field. Explain how to define the Lie derivative $\mathcal{L}_X T$.

(ii) Explain how to define coordinates $(t, x^1, x^2, \ldots)$ such that $X = \partial/\partial t$. State how the Lie derivative acts in these coordinates.

(iii) Prove that $\mathcal{L}_X f = X(f)$ and $\mathcal{L}_X Y = [X, Y]$ where f is a function and Y is a vector field.

(iv) Let T be a tensor of type $(0,2)$. Prove that, in any coordinate basis,

$$(\mathcal{L}_X T)_{\mu\nu} = X^\rho T_{\mu\nu,\rho} + T_{\mu\rho} X^\rho{}_{,\nu} + T_{\rho\nu} X^\rho{}_{,\mu}$$

[*You may assume that the Lie derivative satisfies the Leibnitz rule.*]

(b) The following metric describes a nonlinear gravitational plane wave

$$ds^2 = a(u)(x^2 - y^2)du^2 + 2du dw + dx^2 + dy^2$$

where $a(u)$ is a smooth function.

(i) Write down a Killing vector field of this metric.

(ii) Consider the following vector field

$$X = x f(u) \frac{\partial}{\partial w} + p(u) \frac{\partial}{\partial x}$$

Determine the necessary and sufficient conditions on f and p for X to be a Killing vector field. Deduce that there is a 2-parameter family of Killing vector fields of the above form.

(iii) Find another 2-parameter family of Killing vector fields (this does not require a long calculation).

(iv) Use these results to show that if r, s are points on a surface of constant u then there exists an isometry that maps r to s.

2

(a) A spacetime describing a weak gravitational field has metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{00} = \mathcal{O}(\epsilon^2)$, $h_{0i} = \mathcal{O}(\epsilon^3)$ and $h_{ij} = \mathcal{O}(\epsilon^2)$ where $|\epsilon| \ll 1$.

(i) Explain how the equation of motion of a test body in this spacetime reduces to the Newtonian equation of motion for a test body in a gravitational field, stating clearly any additional assumptions that you make.

(ii) Assume that the matter producing the gravitational field is a perfect fluid with energy-momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$, and the fluid satisfies the non-relativistic conditions $p/\rho = \mathcal{O}(\epsilon^2)$ and $u^i = \mathcal{O}(\epsilon)$. Explain how to recover the Poisson equation of Newtonian gravity. State clearly any further assumptions that you make.

[*In harmonic gauge, the linearized Einstein equation is* $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$.]

(b) A star undergoing a supernova explosion has energy density $T_{00}(t, \mathbf{x})$ in "almost inertial" coordinates. This can be expanded as a sum over spherical harmonics. One way of writing this sum is

$$T_{00}(t, \mathbf{x}) = \sum_{\ell=0}^{\infty} a_{i_1 i_2 \dots i_\ell}(t, r)\hat{x}_{i_1}\hat{x}_{i_2}\dots\hat{x}_{i_\ell}$$

where $r = |\mathbf{x}|$, $\hat{\mathbf{x}} = \mathbf{x}/r$, and $a_{i_1 i_2 \dots i_\ell}(t, r)$ is totally symmetric and traceless on any pair of indices.

Determine the power of the gravitational waves produced by the supernova crossing a large sphere $|\mathbf{x}| = R$ at time t. Simplify your answer as much as possible.

[*You may assume that any isotropic Cartesian tensor is a linear combination of products of* δ_{ij} *and* ϵ_{ijk} *factors.*]

3

The *Kasner metric* is

$$ds^2 = -dt^2 + t^{2p_1}(dx^1)^2 + t^{2p_2}(dx^2)^2 + t^{2p_3}(dx^3)^2$$

where p_1, p_2, p_3 are real constants, not all zero.

Throughout this question, indices i, j, k *take values in* $\{1, 2, 3\}$ *and the summation convention will not be used for these indices.*

(a) Show that the following vector fields form an orthonormal basis:

$$e_0 = \frac{\partial}{\partial t} \qquad\qquad e_i = t^{-p_i}\frac{\partial}{\partial x^i}$$

(b) Write down the corresponding dual basis of 1-forms e^μ.

(c) Determine the connection 1-forms using $de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$.

(d) Determine the curvature 2-forms using $\Theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$. Hence calculate the Riemann tensor components using $\Theta_{\mu\nu} = \frac{1}{2}R_{\mu\nu\rho\sigma}e^\rho \wedge e^\sigma$.

(e) Calculate the Ricci tensor and hence show that the vacuum Einstein equation is satisfied if, and only if,

$$\sum_{i=1}^{3} p_i = A \qquad\qquad \sum_{i=1}^{3} p_i^2 = B$$

for constants A, B whose values you should determine.

4

(a)(i) For a variation of the metric $g_{ab} \to g_{ab} + \delta g_{ab}$ derive formulae for the variations of the volume form, the inverse metric and the Christoffel symbols.

(ii) Show that $\delta R = -R^{ab}\delta g_{ab} + \nabla_a X^a$ for some vector field X^a.

[*In a coordinate basis* $R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}$]

(b) A theory of gravity coupled to a scalar field Φ and Maxwell field F_{ab} has action

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - \frac{1}{2}g^{ab}\nabla_a\Phi\nabla_b\Phi - e^{\alpha\Phi}g^{ac}g^{bd}F_{ab}F_{cd}\right)$$

where α is a constant. The Maxwell field is related to a 1-form potential A_a by $F = dA$.

(i) Derive the equations of motion of the scalar field and Maxwell field.

(ii) Determine the energy-momentum tensor of this theory.

(iii) Prove that this theory satisfies the *null energy condition* $T_{ab}k^a k^b \geqslant 0$ for any null vector k^a.

END OF PAPER