UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 25 June, 2021 12:00 pm to 2:00 pm

PAPER 308

CLASSICAL AND QUANTUM SOLITONS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the scalar field theory in one spatial dimension with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi)\,,$$

where the potential U has the form

$$U(\phi) = \frac{1}{2}(1-\phi^2)^2 + 2\varepsilon\left(\phi - \frac{1}{3}\phi^3\right)$$

with $|\varepsilon| < 1$. Show that U has local minima at $\phi = \pm 1$.

In the case $\varepsilon = 0$, derive the first-order equation satisfied by a static kink solution. Find the explicit kink solution and its energy.

Assume now that ε is small and positive. Find the acceleration of a field configuration that initially has the form of a kink at rest. Show also that there is a static antikink-kink solution (satisfying $\phi \to 1$ as $x \to \pm\infty$) and estimate the separation of the antikink and kink.

[You may assume that when $\varepsilon = 0$ the force exerted by an antikink on a kink is $32e^{-2s}$ where s is the antikink-kink separation.]

2

Explain what is meant by the N-vortex moduli space $\mathcal{M}_N$ for abelian Higgs vortices at critical coupling, and how $\mathcal{M}_N$ acquires a Riemannian metric from the field theory. Describe the geometry of the moduli spaces for $N = 1$ and $N = 2$.

Consider the scattering of two vortices in the centre of mass frame, using the geodesic approximation to scattering. Describe qualitatively how the scattering angle varies with impact parameter. Compare this scattering behaviour of vortices with the scattering behaviour of two identical point particles in two dimensions, subject to a repulsive central potential that has a finite maximum when the particles coincide. Comment on the difference between the range of scattering angles that occur.

[In each case you should sketch a graph of scattering angle versus impact parameter, for impact parameters running from $-\infty$ to ∞, and for two fixed values of the incoming speeds — one small and one large.]

3

Give an account of the rational map approximation for Skyrmions, mentioning some of its topological and energetic aspects.

There is an icosahedrally symmetric Skyrmion with baryon number 7 that can be approximated using the rational map

$$R(z) = \frac{z^7 - 7z^2}{7z^5 + 1}.$$

Verify that $R(z)$ has C_5 symmetry around the point $z = 0$, and that this extends to D_{5d} symmetry.

Find the Wronskian of $R(z)$ and hence the zeros of the Wronskian. Sketch the locations of the zeros in the complex plane. Interpret these locations geometrically and physically in the context of the Skyrmion.

END OF PAPER