UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 24 June, 2021 12:00 pm to 2:00 pm

PAPER 307

SUPERSYMMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the action

$$S[x,\psi] = \int \left[\frac{1}{2}\dot{x}^2 + i\bar{\psi}\dot{\psi} - \frac{1}{2}(\partial h)^2 + \bar{\psi}\psi\, \partial^2 h \right] dt$$

where $x(t)$ is a real bosonic variable, $\psi(t)$ is a complex fermionic variable with complex conjugate $\bar{\psi}(t)$, and h is a smooth, real function of x.

Show that, up to boundary terms, $S[x,\psi]$ is invariant under complex conjugation and under the supersymmetry transformations

$$\begin{aligned} \delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi \\ \delta\psi &= \epsilon(i\dot{x} - \partial h) \\ \delta\bar{\psi} &= -\bar{\epsilon}(i\dot{x} + \partial h) \end{aligned}$$

and find the Noether charges Q and $\bar{Q}$ corresponding to the parameters $\bar{\epsilon}$ and ϵ, respectively.

We now wish to quantize this theory. What are the canonical commutation relations among the various fields? Explain why the Hilbert space $\mathcal{H}$ can be represented as the space of square-integrable, complex valued forms on $\mathbb{R}$, and identify how Q and $\bar{Q}$ act on $\mathcal{H}$.

Find the unique ground state in the case $h(x) = -x^4$ *[You need not determine the overall constant of normalization, provided it is finite.]* Show there is no normalizable state of zero energy when $h(x)$ is a generic cubic polynomial.

2 Consider an $\mathcal{N} = (2,2)$ supersymmetric $U(1)$ gauge theory in $d = 2$. Let $\Phi^i(x^\pm, \theta^\pm, \bar{\theta}^\pm)$ be a chiral superfield of charge q_i and let $V(x^\pm, \theta^\pm, \bar{\theta}^\pm)$ be the vector supermultiplet containing the photon $A_\mu(x^\pm)$.

Write down how gauge transformations act on Φ^i, $\bar{\Phi}^{\bar{\imath}}$ and V. Explain what is meant by *Wess-Zumino gauge*. Define the *fieldstrength supermultiplet* Σ and show that it is a gauge invariant twisted chiral superfield. What axial charge must be assigned to Σ if the Fayet-Iliopoulous term

$$\frac{t}{2}\int \Sigma \; d\theta^+ d\bar{\theta}^- d^2x$$

is to be invariant under axial transformations?

[*You may use the fact that the chiral derivatives obey* $\{\bar{D}_+, D_-\} = 0$ *and* $\bar{D}_+^2 = 0 = D_-^2$ *without proof.*]

Consider the theory on a worldsheet torus with action

$$S[\Phi^i, V] = \int \sum_i \bar{\Phi}^{\bar{\imath}} e^{q_i V} \Phi^i \; d^4\theta \, d^2x - \left(\frac{t}{2} \int \Sigma \; d\theta^+ d\bar{\theta}^- d^2x \; + \; \text{c.c.} \right)$$

in which the gauge supermultiplet V is taken to be a non-dynamical, background field. Explain why the axial U(1) transformations are preserved at the quantum level only if $\sum_i q_i = 0$.

3 In two bosonic dimensions, (0,2) superspace is defined as the surface $\theta^- = \bar{\theta}^- = 0$ inside (2,2) superspace. A (0,2) chiral field is a superfield Φ that obeys $\bar{D}_+\Phi = 0$ where

$$\bar{D}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\frac{\partial}{\partial x^+}$$

is the chiral derivative. What is the component expansion of a (0,2) chiral superfield? Construct a supersymmetric action $S_1[\Phi]$ whose component expansion includes the standard kinetic term for a complex scalar field.

Now let Λ_- be a (0,2) superfield whose leading component field is a fermionic spinor λ_-, and let Λ_- obey $\bar{D}_+\Lambda_- = f(\Phi^i)$, where f is a holomorphic function of chiral superfields Φ^i. Find the component expansion of Λ_- and the component field expression for the action

$$S_2 = -\frac{1}{2}\int \bar{\Lambda}_-\Lambda_- \; d\theta^+ d\bar{\theta}^+ \, d^2x \, .$$

Identify which components of Λ are auxiliary.

In a theory of several such fermionic superfields Λ_{-a}, each obeying $\bar{D}_+\Lambda_{-a} = f_a(\Phi^i)$, it is desired to add the terms

$$S_3 = -\left(\int \Lambda_{-a} J^a(\Phi^i) \; d\theta^+ d^2y \;\; + \;\; \text{complex conjugate}\right)$$

to the action, where $J^a(\Phi^i)$ are further holomorphic functions of (0,2) chiral superfields and $y^+ = x^+ + i\theta^+\bar{\theta}^+$ while $y^- = x^-$. Under what condition(s) are these terms also supersymmetric?

Find the potential experienced by the scalars ϕ^i in the theory $S_1 + S_2 + S_3$ after eliminating the auxiliary field(s).

END OF PAPER