UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 18 June, 2021 12:00 pm to 3:00 pm

PAPER 306

STRING THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Polyakov action for the closed string in flat spacetime, with metric $\eta_{\mu\nu}$, is given by

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{|h|}h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu.$$

1. By considering the variation of the action with respect to the worldsheet metric h_{ab}, find a classical expression for the stress tensor T_{ab}. Show that the stress tensor is traceless.

2. Write down an expression for the Nambu-Goto action for the closed string. Show that the Nambu-Goto and Polyakov actions are classically equivalent.

3. Consider the path integral expression for the correlation function

$$\langle\phi_1...\phi_n\rangle = \mathcal{N}\int \mathcal{D}X\phi_1...\phi_n e^{iS_h[X]}, \tag{1}$$

where ϕ_i are n local fields on the worldsheet, $\mathcal{N}$ is a normalization constant and $S_h[X]$ is the Polyakov action with the worldsheet metric fixed to a particular gauge. Assuming that the fields ϕ_i are invariant under Weyl transformations, show that the correlation function (1) is Weyl-invariant if

$$h^{ab}\langle T_{ab}\phi_1...\phi_n\rangle = 0.$$

4. Now consider a change in the background metric of the target space

$$\eta_{\mu\nu}\to\eta_{\mu\nu}+\varepsilon_{\mu\nu},$$

where $\varepsilon_{\mu\nu}$ is a constant symmetric tensor. Show that the change in the correlation function (1) is given, to leading order, by

$$\int_\Sigma d^2\sigma\langle\mathcal{O}(\sigma,\tau)\phi_1...\phi_n\rangle,$$

where $\mathcal{O}(\sigma,\tau)$ is a field you should find. Briefly explain why a change in the worldsheet metric h_{ab} should not change the correlation functions, whereas a change in the target space metric can change the correlation functions.

2

After gauge fixing, the the left-moving sector of closed string may be described in terms of the embedding fields $X^\mu(z)$ and ghost fields $b(z)$ and $c(z)$, where z is the left-moving worldsheet coordinate.

1. The stress tensor for the embedding fields is given by
$$T_X(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu(z) : .$$
Under infinitesimal conformal transformations, given by a holomorphic vector field $v(z)$, $T_X(z)$ transforms as
$$\delta_v T_X(z) = 2\partial v(z) T_X(z) + v(z)\partial T_X(z) + c_X \partial^3 v(z).$$
By considering the OPE of $T_X(z)$ and $T_X(w)$, calculate the value of the constant c_X. [Hint: You do not need to compute the full $T_X(z)T_X(w)$ OPE.]

2. The ghost stress tensor is given by
$$T_{\rm gh}(z) = - : \partial b(z) c(z) : -2 : b(z)\partial c(z) : .$$
Assuming the form of the transformation of the ghost stress tensor is
$$\delta_v T_{\rm gh}(z) = 2\partial v(z) T_{\rm gh}(z) + v(z)\partial T_{\rm gh}(z) + c_{\rm gh}\partial^3 v(z),$$
calculate the value of the constant $c_{\rm gh}$.

3. Defining the total stress tensor to be
$$T_{\rm tot}(z) = T_X(z) + T_{\rm gh}(z),$$
determine how the total stress tensor transforms under infinitesimal conformal transformations. Hence derive the commutator $[\mathcal{L}_m, \mathcal{L}_n]$, where $\mathcal{L}_n$ are the modes of the total stress tensor. What happens when the dimension of spacetime is 26?

3

The closed string tachyon has mass $M^2 = -4/\alpha'$ and the integrated and unintegrated closed string tachyon vertex operators are

$$V = \int d^2z : e^{ik\cdot X(z,\bar{z})} :, \qquad U(z,\bar{z}) =: c(z)\bar{c}(\bar{z})e^{ik\cdot X(z,\bar{z})} : .$$

The scattering amplitude of n tachyons at tree level may be written as

$$\mathcal{A}_n = g_c^{n-2}\Big\langle U(z_1,\bar{z}_1)U(z_2,\bar{z}_2)U(z_3,\bar{z}_3)V_4...V_n\Big\rangle,$$

where g_c is the closed string coupling.

1. Briefly explain why $\mathcal{A}_n$ is independent of the coordinates $(z_i, \bar{z}_i)$.
2. Show that the form of the closed string scattering amplitude of n tachyons is given by
$$\mathcal{A}_n = \delta^{26}\left(\sum_{i=1}^{n} k_i^\mu\right)\int d^2z_4...\int d^2z_n\,\mathcal{F}_n,$$
where $\mathcal{F}_n$ is a function of the z_i and k_i^μ which you should find. You may assume
$$\langle c(z_1)\bar{c}(\bar{z}_1)c(z_2)\bar{c}(\bar{z}_2)c(z_3)\bar{c}(\bar{z}_3)\rangle = |z_1 - z_2|^2|z_2 - z_3|^2|z_3 - z_1|^2.$$
3. the Mandelstam variables s, t and u are given by
$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2.$$
By taking $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ and setting $z_4 \equiv z$ show that the four-point amplitude may be written as
$$\mathcal{A}_4 = \delta^{26}\left(\sum_{i=1}^{n} k_i^\mu\right) g_c^2 \int d^2z\; |z|^{-\alpha' u/2-4}|1-z|^{-\alpha' t/2-4}.$$
By choosing different values for the z_i, explain why the amplitude $\mathcal{A}_4$ is invariant under the exchange of $s \leftrightarrow t$.

4

1. An open string has endpoints at $\sigma = 0, \pi$. Starting with the gauge-fixed Polyakov action
$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\, \eta_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu},$$
derive the two possible boundary conditions consistent with the equations of motion.

2. Find expressions for the embedding fields $X^{\mu}(\sigma,\tau)$ and $X^{I}(\sigma,\tau)$ of the string in the presence of two parallel D-branes, the worldvolumes of which extend in the $\mu = 0, 1, .., p$ directions. The D-branes are seperated by a distance $x_2^I - x_1^I$ in the remaining $D - p - 1$ directions, where $I = p + 1, ..., D$.

3. Show that the masses M of the open strings described by the above configuration are given by
$$M^2 = T^2|x_2 - x_1|^2 + \frac{1}{\alpha'}(N-1),$$
where T and N should be found. Describe the massless spectrum of the string and the interpretation of the massless states from the perspective of the D-branes worldvolumes. What happens to the gauge symmetry on the worldvolumes of the D-branes when the D-branes coincide?

END OF PAPER