UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 17 June, 2021 12:00 pm to 3:00 pm

PAPER 305

THE STANDARD MODEL

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a vector field $X_\mu(x)$ of mass $M_X \neq 0$.

(a) By quoting results for the representations of the Poincaré group, determine the number of degrees of freedom in X_μ defined by the number of polarisations of the corresponding one-particle states and show explicitly the difference with the massless case.

(b) Derive the propagator in momentum space for X_μ.

(c) Contrary to the Coulomb interactions $V \sim 1/r$ which are long range, show that this field being massive, mediates short range interactions (you may consider for simplicity the case of a massive scalar which is equivalent). Could you identify another example of short range interactions but mediated by a massless ($M_X = 0$) vector field? Explain.

(d) If X_μ is originally a gauge field that becomes massive from the Higgs mechanism, briefly determine how it couples to fermions charged under the corresponding gauge symmetry and then show that these interactions reduce to the 4-Fermi interaction at energies smaller than the X_μ mass, M_X. Determine the Fermi coupling in terms of M_X.

(e) Describe the potential problem that a field theory of a massive vector field has regarding the breakdown of perturbative unitarity at high energies and outline how the presence of the Higgs field and its couplings can solve this problem.

2

(a) Define what is meant by an anomaly in field theory and explain the difference between anomalies for global and local symmetries.

(b) Consider a $U(1)$ gauge theory for massless fermions ψ of charge e. Show that besides the conserved gauge current there is a classically conserved axial current $J^\mu_{ax} = \bar{\psi}\gamma^\mu\gamma^5\psi$ corresponding to the global $U(1)_{ax}$ transformation $\psi \to e^{\mathrm{i}\beta\gamma_5}\psi$. Derive the expression for the $U(1)_{ax}$ anomaly and show that quantum mechanically the axial current change can be written as:

$$\partial_\mu J^\mu_{ax} = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$$

where $F_{\mu\nu}$ is the $U(1)$ gauge field strength. *[You may find useful the identity* $\mathrm{Tr}\left\{\gamma_5\left[\gamma_\mu,\gamma_\nu\right]\left[\gamma_\rho,\gamma_\sigma\right]\right\} = 16\,\mathrm{i}\,\epsilon_{\mu\nu\rho\sigma}$*].*

(c) Show that the gauge symmetries of the Standard Model are free from anomalies. Determine why anomaly cancelation explains that the electric charges of the electron and the proton are equal and opposite.

[Recall that the quantum numbers under $SU(3)_c \times SU(2)_L \times U(1)_Y$ for a family of matter fields are $\left(\mathbf{3},\mathbf{2},\frac{1}{6}\right) + \left(\bar{\mathbf{3}},\mathbf{1},\frac{2}{3}\right) + \left(\bar{\mathbf{3}},\mathbf{1},-\frac{1}{3}\right) + \left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right) + (\mathbf{1},\mathbf{1},\,-1) + (\mathbf{1},\mathbf{1},0)$.]

3

(a) State and prove Goldstone's theorem using both classical and quantum arguments (you may use without proof that the conserved charges Q^a are quantum operators such that $[\phi_i, Q^a] = \mathrm{i}T^a_{ij}\phi_j$ with ϕ_i scalar fields transforming in the representation of the algebra generators T^a_{ij}).

(b) Consider an $SU(3)$ gauge theory with gauge field $A_\mu(x)$ coupled to a scalar field Φ in the fundamental representation.

(i) State how $A_\mu(x)$ transforms under a gauge transformation and prove that the corresponding field strength $F_{\mu\nu}(x)$ is gauge covariant.

(ii) Write down the most general renormalisable Lagrangian determining the couplings of Φ to the gauge fields, including kinetic terms for both fields and the scalar potential for Φ.

(iii) Show that the symmetry may be broken by a non-vanishing vacuum expectation value for Φ. Determine the pattern of symmetry breaking and verify Goldstone's theorem.

(iv) Show explicitly how the Goldstone modes become the longitudinal modes of the original gauge field to give it a mass. Are there other massive states in this theory?

4

(a) By considering the couplings of the quark fields to the $W^{\pm}$ gauge bosons and to the Higgs boson (Yukawa couplings), explain the origin of the Cabibbo-Kobayashi-Maskawa (CKM) matrix and determine its minimum number of free parameters.

(b) Show that the presence of a phase in the CKM matrix implies that CP is broken in electroweak interactions.

(c) Show that the $\theta\epsilon^{\mu\nu\rho\sigma}G^A_{\mu\nu}G^A_{\rho\sigma}$ term in the QCD Lagrangian also breaks CP and T but not CPT where $G^A_{\mu\nu}$ is the field strength of the QCD gauge field.

(d) Explain clearly what is meant by the strong CP problem and outline one potential solution to this problem.

(e) Explain why the θ parameter for the weak interactions can always be rotated away.

END OF PAPER