UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 21 June, 2021 12:00 pm to 3:00 pm

## PAPER 304

## ADVANCED QUANTUM FIELD THEORY

***Before you begin please read these instructions carefully***

*Candidates have THREE HOURS to complete the written examination.*

*Attempt no more than* ***THREE*** *questions.*
*There are* ***FOUR*** *questions in total.*
*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Consider a theory of $N$ real scalar fields $\phi_a$ ($a = 1, \ldots, N$) in $d > 0$ dimensions, with classical action $S[\phi]$, say in Euclidean spacetime. (We adopt the notation that $\phi = (\phi_1, \ldots, \phi_N)$.) Write down the path integral expression for the generating functional $\mathcal{Z}[J]$, explaining the meaning of $J$ and why $\mathcal{Z}[J]$ is called the generating functional. That is, what does $\mathcal{Z}[J]$ generate and how?

Given the Wilsonian effective action $W[J] = -\log \mathcal{Z}[J]$, write down the quantum effective action $\Gamma[\Phi]$ in terms of $W[J]$, being sure to define $\Phi$.

Consider the first and second functional derivatives of $\Gamma[\Phi]$ and interpret how they are related to quantities discussed above.

Given that $\mathcal{Z}[J]$ may be expressed in a perturbative expansion as a sum of Feynman diagrams, how can $W[J]$ and $\Gamma[\Phi]$ be expressed in perturbative expansions? Justify your answers mathematically.

Assume that the classical action and the path integral measure are invariant under the field transformation

$$\phi_a(x) \mapsto \phi'_a(x) = U_{ab}\phi_b(x)$$

where $U$ is a constant $N \times N$ invertible matrix. Show that the generating functional $\mathcal{Z}[J]$ is invariant under the transformation

$$J_a(x) \mapsto J'_a(x) = J_b(x)U_{ba}\,.$$

Use this to show that the quantum effective action $\Gamma[\Phi]$ is invariant under

$$\Phi_a(x) \mapsto \Phi'_a(x) = U_{ab}\Phi_b(x)\,.$$

**2**

Consider a theory of two real, scalar fields, $\phi_0$ and $\chi_0$, in $d$ Euclidean dimensions with the action

$$S = \int d^d x \left[ \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{2}(\partial\chi_0)^2 + \frac{1}{2} M_0^2 \chi_0^2 + \frac{1}{2}\lambda_0 \phi_0 \chi_0^2 \right] .$$

Write down the momentum space Feynman rules.

Working at one-loop order in perturbation theory, draw all one-particle irreducible (1PI) diagrams with 1, 2, or 3 external legs.

Explain how the exact, connected two-point function

$$\tilde{G}^{(2)}(p) = \int d^d x \, e^{-ip\cdot x} \langle \phi(x)\phi(0) \rangle^{\rm conn}$$

is related to the $\phi$ self-energy $\Pi(p^2)$, the sum of all amputated, 1PI two-point functions with $\phi$ external legs.

Using dimensional regularization with $d = 6 - \epsilon$ (and $\epsilon$ small), show that the $\phi$ self-energy at one-loop order is equal to

$$\Pi_1(p^2) = \frac{1}{\epsilon}(A + Bp^2) + C(p^2, \mu)$$

where you should determine the constants $A$ and $B$, and the function $C$. $C$ is a function of $p^2$ and the renormalization scale $\mu$, and it remains finite as $\epsilon \to 0$. [You can leave terms in $C$ in integral form.]

What steps are taken to obtain finite results for the one-loop contribution to $\tilde{G}^{(2)}(p)$? How is the renormalized $\phi$ mass $m(\mu)$ related to the physical $\phi$ mass $m_{\rm phys}$ in the minimal subtraction scheme?

Briefly explain the meaning of the phrase *superficial degree of divergence* in relation to loop diagrams in this theory.

In addition to what you found in the calculation of the $\phi$ self-energy, what other counterterms are needed in order to arrive at a fully renormalized action? [For this part, explain what calculations need to be done, without explicitly doing them.]

[*Hints: You may use the following result for d-dimensional integrals with integer a and b:*

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{(\ell^2)^a}{(\ell^2 + \Delta)^b} = \frac{\Gamma(b - a - \frac{d}{2})\,\Gamma(a + \frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(b)\Gamma(d/2)} \, \Delta^{-(b-a-d/2)}$$

*as well as the Laurent expansion of the* $\Gamma$*-function near its pole at* $z = 0$*:*

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z)$$

*where* $\gamma$ *is the Euler–Mascheroni constant. Also recall that* $(z-1)\Gamma(z-1) = \Gamma(z)$*.* ]

**3**

Consider a field theory for a single scalar field which has a single dimensionless coupling $g$ and no mass parameters. Let $G^{(n)}(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle^{\text{conn}}$ be the renormalized, connected $n$-point correlation function for the scalar field $\phi$. Explain why this correlation function must also depend on an additional scale $\mu$ and obey the following equation

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right) G^{(n)}(x_1, \ldots, x_n) = 0 \,, \qquad (*)$$

where you should define the functions $\beta(g)$ and $\gamma(g)$.

What is the significance of the $\beta$-function and of any fixed points $g_*$ of $\beta(g)$? What does the sign of the $\beta$-function tell us about the behaviour of the coupling as $\mu$ is increased from some value $\mu_0$?

Take as given that the renormalized propagator obeys

$$\int d^4x \, e^{ip \cdot x} \, G^{(2)}(x, 0) = \frac{1}{p^2} \, C\left( \frac{p^2}{\mu^2}, g(\mu) \right)$$

for some function $C$. In the case $n = 2$, obtain a solution of $(*)$ in the form

$$C\left( \frac{p^2}{\mu^2}, g(\mu) \right) = f(\mu) \, C\left( \frac{p^2}{\mu_0^2}, g(\mu_0) \right)$$

for arbitrary scale $\mu_0$ and running coupling $g(\mu)$, the functional form of which is unspecified here. You should show that the function $f(\mu)$ takes the form $f(\mu) = \exp(h(\mu))$ where

$$h(\mu) \propto \int_{g(\mu_0)}^{g(\mu)} \frac{\gamma(g)}{\beta(g)} \, dg \,.$$

[*Hint: It may be helpful to use the substitution* $t = \log(\mu/\mu_0)$.]

Now assume that $\beta(g) = -bg^3$ with $b > 0$. Find all fixed points of the $\beta$-function and describe their nature. Express $g(\mu)$ in terms of $g(\mu_0)$ and the ratio $\mu/\mu_0$. Let $\Lambda$ be the scale at which $g(\mu)$ diverges according to this expression. Write $g(\mu)$ in terms of $\mu$ and $\Lambda$.

Assuming further that $\gamma(g) = cg^2$, evaluate

$$\frac{C(p^2/\mu^2, g(\mu))}{C(1, g(p))} \,.$$

**4**

Working in 4 Euclidean dimensions, consider an $SU(2)$ gauge theory containing a Dirac fermion field $\psi$ (mass $m$) which transforms in the fundamental representation and a scalar field $\phi = \phi^a T^a$ (mass $M$) which transforms in the adjoint representation. That is, under a gauge transformation

$$\begin{aligned}\psi(x) &\mapsto e^{i\alpha^a(x)T^a}\,\psi(x)\\ \bar{\psi}(x) &\mapsto \bar{\psi}(x)\,e^{-i\alpha^a(x)T^a}\\ \phi(x) &\mapsto e^{i\alpha^a(x)T^a}\,\phi(x)\,e^{-i\alpha^b(x)T^b}\,.\end{aligned}$$

[Recall that the $SU(2)$ generators in the fundamental representation are related to the Pauli matrices by $T^a = \frac{1}{2}\sigma^a$, and the generators in the adjoint representation are related to the Levi-Civita antisymmetric tensor $(T^a_{\text{adj}})^{bc} = -i\varepsilon^{abc}$. You may use without proof the fact that the adjoint representation is real.]

(a) Construct gauge-covariant derivatives for the fermion and scalar fields, specifying how the gauge fields must transform under gauge transformations.

(b) Write down the gauge-invariant Lagrangian containing all renormalizable terms, being sure to demonstrate the gauge-invariance of each term.

(c) Briefly explain in a few sentences why one would fix gauge and why the gauge-fixed Lagrangian might contain ghost fields and an auxiliary Nakanishi-Lautrup field.

(d) BRST transformations acting on gauge ($A^a_\mu$), ghost ($c^a$ and $\bar{c}^a$), and the auxiliary Nakanishi-Lautrup ($B^a$) fields are defined as

$$\delta_B A^a_\mu = \eta(D_\mu c)^a \qquad\qquad \delta_B c^a = -\frac{\eta g}{2}\varepsilon^{abc}c^b c^c$$
$$\delta_B \bar{c}^a = \eta B^a \qquad\qquad \delta_B B^a = 0\,,$$

where $\eta$ is a Grassmann-valued parameter. Show that these transformations are nilpotent when acting on any operator $O(A_\mu, c, \bar{c}, B)$ which is polynomial in those fields. [*Hint: Consider two successive transformations with respective parameters* $\eta_1$ *and* $\eta_2$.]

## END OF PAPER