UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 14 June, 2021 12:00 pm to 2:00 pm

PAPER 303

STATISTICAL FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a theory which has an effective free energy

$$f(T,m) = a_2(T)\, m^2 + a_4\, m^4 + a_6\, m^6\,,$$

in the mean field approximation, where m is the magnetisation, $a_6 > 0$ and $a_2(T)$ varies from positive to negative as the temperature T is lowered.

(a) For given a_2, a_4, a_6 and T, how can the equilibrium value of m be determined from f?

(b) If $a_4 < 0$, sketch $f(m)$ at the temperature where there is a first order phase transition. Find an expression for a_2 in terms of a_4 and a_6 at this point. What is the jump in the magnetisation at this phase transition?

(c) If $a_4 = 0$, a phase transition occurs at $a_2(T_c) = 0$ and $a_2 \sim (T - T_c)$ in the vicinity of this point. Compute the critical exponents α, β, γ and δ at this phase transition.

(d) Why would you expect the critical exponents calculated in part (c) to be inaccurate when the number of dimensions is small?

(e) Consider a model defined on a square lattice in d dimensions. The N lattice sites are labelled by i and the spin variable σ_i at site i takes the value 1, 0, or -1. The energy is given by

$$E = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j + g \sum_i \sigma_i^2 - B \sum_i \sigma_i\,,$$

where $J > 0$, g and the magnetic field B are constants, and $\langle ij \rangle$ means that the sum is over nearest neighbour pairs.

(i) Using the mean field approach, show that the effective free energy per unit site can be approximated by

$$f = \frac{F}{N} = \frac{1}{2} Jqm^2 - T \ln\Big[1 + 2\kappa \cosh\big(\beta(Jqm + B)\big)\Big]\,,$$

where $\beta = 1/T$, $\kappa = \exp(-\beta g)$ and q is the number of nearest neighbours of each site. [*Hint: Do not approximate the* $g \sum_i \sigma_i^2$ *term.*]

(ii) For $B = 0$, by treating m^2 as small and expanding f as a power series in m^2, find the values of T and κ for which the mean field approximation predicts a tri-critical point.

2

Consider a theory involving a real scalar field ϕ in d dimensions with a free energy of the form

$$F[\phi] = \int d^dx \Big[\tfrac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + \tfrac{1}{2}\mu_0^2\,\phi^2 + \ldots \Big] . \qquad (*)$$

(a) Describe the three steps of the renormalisation group procedure (in momentum space) for such a free energy and explain how they result in a flow of the parameters in the free energy.

(b) Calculate the naive (engineering) dimension of ϕ.

(c) Now suppose that, close to the relevant fixed point, the correlation function scales like

$$\langle \phi(\boldsymbol{x})\phi(0) \rangle \sim \frac{1}{|\boldsymbol{x}|^{d-2+\eta}} ,$$

for some constant η. Calculate the scaling dimension Δ_ϕ of the field ϕ in terms of η. Why does this differ from the engineering dimension?

(d) Ignoring interactions, compute the scaling dimension Δ_g of the coupling g appearing in a term in the free energy

$$\sim \int d^dx\, g\, \phi^n \,(\nabla^2\phi)^m ,$$

where n and m are non-negative integers. Give conditions on Δ_g for the coupling to be relevant, marginal and irrelevant.

(e) Suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with

$$\sim \int d^dx \Big[\alpha_0\,\phi^3 + \gamma_0\,\phi^5 \Big] .$$

Compute the lowest order correction to the coupling α_0 from γ_0. Draw a Feynman diagram for this correction. [You may assume that $\langle \phi^+_{\boldsymbol{k}}\,\phi^+_{\boldsymbol{k}'} \rangle_+ = (2\pi)^d\,\delta^{(d)}(\boldsymbol{k}+\boldsymbol{k}')G_0(k)$, where $G_0(k) = 1/(k^2+\mu_0^2)$, for appropriately defined ϕ^+ and $\langle \ldots \rangle_+$, and you may leave your final answer in integral form.]

(f) Suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with

$$\sim \int d^dx \Big[\alpha_0\,\phi^3 + \lambda_0\,\phi^4 \Big] .$$

(i) What is the lowest order at which α_0 appears in a correction to λ_0? [You are not required to compute the correction.]

(ii) What is the lowest order at which λ_0 appears in a correction to α_0? [You are not required to compute the correction.]

3

Consider an N-component real field $\boldsymbol{n}(\boldsymbol{x})$ which is constrained to have unit length, $\boldsymbol{n} \cdot \boldsymbol{n} = 1$, in d dimensions. The free energy is

$$F[\boldsymbol{n}] = \int d^d x \, \frac{1}{2g} (\partial_i n_A)(\partial_i n_A) \, ,$$

where repeated indices are summed over, $i = 1, 2, \ldots, d$ and $A = 1, 2, \ldots, N$.

(a) What is the naive (engineering) dimension of the coupling g?

(b) By writing $\boldsymbol{n}(\boldsymbol{x}) = (\boldsymbol{\pi}(\boldsymbol{x}), \sigma(\boldsymbol{x}))$, where $\boldsymbol{\pi}$ has $N-1$ components, show that the free energy can be expressed as

$$F[\boldsymbol{\pi}] = \int d^d x \, \frac{1}{2g} \left[(\partial_i \pi_a)(\partial_i \pi_a) + \frac{\pi_a (\partial_i \pi_a) \pi_b (\partial_i \pi_b)}{1 - \boldsymbol{\pi} \cdot \boldsymbol{\pi}} \right] ,$$

where repeated indices are summed over, $i = 1, 2, \ldots, d$ and $a, b = 1, 2, \ldots, N-1$.

(c) By Taylor expanding for small $\boldsymbol{\pi}$ and integrating out wavevectors over an interval Λ/ζ to Λ, show that, to leading order,

$$\frac{1}{g(\zeta)} = \zeta^{d-2} \left(\frac{1}{g_0} + (2 - N) I_d \right) ,$$

where

$$I_d = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \, .$$

[You may assume that $\langle \pi^+_{\boldsymbol{k}\, a} \, \pi^+_{\boldsymbol{k}'\, b} \rangle_+ = g_0 \, \delta_{ab} \, (2\pi)^d \, \delta^{(d)}(\boldsymbol{k} + \boldsymbol{k}')/k^2$ for appropriately defined π^+ and $\langle \ldots \rangle_+$, and that the appropriate scaling for the fields is $\boldsymbol{\pi}' = \boldsymbol{\pi}^-/A$ where $A = 1 - \frac{1}{2}(N-1) \, g_0 \, I_d$.]

(d) If $d = 2 + \epsilon$ and $\epsilon \geqslant 0$ is small, show that the beta function can be approximated as

$$\frac{dg}{ds} \approx -\epsilon \, g + \frac{\Omega_{d-1}}{(2\pi)^d} \Lambda^{\epsilon} \, (N-2) \, g^2 \approx -\epsilon \, g + (N-2) \, \Lambda^{\epsilon} \, \frac{g^2}{2\pi} \, ,$$

where $s = \ln \zeta$ and Ω_{d-1} is the area of the unit sphere S^{d-1}.

(e) Find the fixed points, $g_\star$, of this equation.

(f) Identifying g with temperature, find the critical exponent ν for these fixed points.

END OF PAPER