UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2021 12:00 pm to 3:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Take a simple Lie algebra L with exactly two simple roots $\boldsymbol{\alpha}_1 = (1,0)$ and $\boldsymbol{\alpha}_2 = (-3/2, \sqrt{3}/2)$. Assume that $E_{i\pm}, H_i$ are the $SU(2)$ generators associated with root $\boldsymbol{\alpha}_i$ $(i = 1,2)$. Here, we define a weight to be positive if its H_2 eigenvalue is positive (if this is zero, then the weight is positive if the eigenvalue of H_1 is positive).

(i) Calculate the Cartan matrix of L.

(ii) State how many times $\boldsymbol{\alpha}_1$ may be raised with $\boldsymbol{\alpha}_2$ without annihilating the state and how many times $\boldsymbol{\alpha}_2$ may be raised with $\boldsymbol{\alpha}_1$ without annihilating the state.

(iii) If $\boldsymbol{\mu}$ is a weight and $\boldsymbol{\alpha}$ a root, write down another weight in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$ assuming Weyl symmetry.

(iv) Thus construct the adjoint representation of L and draw its weight diagram, labelling it carefully.

(v) Determine the fundamental weights $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$.

(vi) By using Weyl symmetry and $E_{i\pm}$, find weights for the irreducible representation with highest weight vector $\boldsymbol{\mu}_1$ and draw the weight diagram.

2 A Lie group has group elements $g(x)$ depending on group parameters x^r, with $g(0) = e$, the identity, and under group multiplication $g(x)g(y) = g(z)$ for $z^r = \varphi^r(x, y)$, a smooth function of x and y. Let $g(x)^{-1} = g(\bar{x})$. Assume general group axioms throughout this question.

(i) Find $\varphi^r(x, 0)$, $\varphi^r(0, x)$ and $\varphi^r(\bar{x}, x)$ in terms of x^r.

(ii) Expanding $\varphi^r(x, y)$ near the origin, we write

$$\varphi^r(x, y) = F^r + Ax^r + By^r + C^r{}_{st}x^s y^t + D^r{}_{st}x^s x^t + E^r{}_{st}y^s y^t + \ldots,$$

where '…' represents terms higher order in the components of x and y. Find the numerical constants F^r, A, B, $D^r{}_{st}$, $E^r{}_{st}$.

(iii) Find $\bar{x}(x)$ for small x^r up to and including terms of order $x^r x^t$.

(iv) Let $g(w) = g(x)^{-1}g(y)^{-1}g(x)g(y)$. For small x and y, find w^r in terms of $C^r{}_{st}$ and the components of x and y up to and including terms of second order in the group parameters.

(v) Consider $g(z + dz) = g(z)g(\theta)$, where $g(z)$ is an arbitrary group element. Find dz^r in terms of

$$\mu_a{}^r(z) = \left.\frac{\partial\varphi^r(z, \theta)}{\partial\theta^a}\right|_{\theta=0}$$

and the infinitesimal parameters θ^a.

(vi) Now take $g(z) = g(x)g(y)$ for fixed $g(x)$ and let $g(y)$ undergo an infinitesimal change, i.e. $g(z + dz) = g(x)g(y + dy)$. Derive $\partial z^r/\partial y^s$ in terms of $\mu_d{}^r(z)$ and $\lambda_s{}^d(y)$, where $[\lambda(y)]$ is the matrix inverse of $[\mu(y)]$.

(vii) Defining $T_a(y) = \mu_a{}^s(y)\partial/\partial y^s$, show that $T_a(y) = T_a(z)$.

(viii) Show that

$$\mu_a{}^s(y)\mu_b{}^t(y)\frac{\partial^2 z^r}{\partial y^s \partial y^t} = \mu_a{}^s(y)[T_b(y)\lambda_s{}^c(y)]\mu_c{}^r(z) + T_b(z)\mu_a{}^r(z).$$

(ix) Thus derive

$$\mu_a{}^s(y)\mu_b{}^t(y)\frac{\partial^2 z^r}{\partial y^s \partial y^t}\lambda_r{}^c(z) = -[T_b(y)\mu_a{}^r(y)]\lambda_r{}^c(y) + [T_b(z)\mu_a{}^r(z)]\lambda_r{}^c(z).$$

(x) By considering symmetry under $a \leftrightarrow b$, deduce that $f^c{}_{ab}$ are constants, where we define $f^c{}_{ab} = [T_a(y)\mu_b{}^r(y) - T_b(y)\mu_a{}^r(y)]\,\lambda_r{}^c(y)$.

(xi) Thereby derive $[T_a, T_b] = f^c{}_{ab}T_c$.

(xii) Using $g(x) = \exp(x^a T_a)$ for a generator T_a satisfying the relation in (xi), use $\exp(tA)\ \exp(tB) = \exp(t(A + B) + t^2[A, B]/2 + \mathcal{O}(t^3))$ to derive $f^c{}_{ab}$ in terms of $C^c{}_{ab}$.

3

(i) By checking properties of M, find three different matrix groups containing elements

$$M = \begin{pmatrix} \frac{f(\theta)}{\sqrt{2}} - i\frac{\sin\theta}{2f(\theta)} & -\frac{\sin\theta}{2f(\theta)} \\ \frac{\sin\theta}{2f(\theta)} & \frac{f(\theta)}{\sqrt{2}} + i\frac{\sin\theta}{2f(\theta)} \end{pmatrix}, \qquad f(\theta) = \sqrt{1+\cos\theta},$$

where $\theta \in \mathbb{R}$.

(ii) Now consider an element of $SO(3)$

$$R_{ab} = \cos\theta\delta_{ab} + (1-\cos\theta)n_a n_b - \sin\theta\epsilon_{abc}n_c,$$

where $a, b, c = 1, 2, 3$, $n_a n_a = 1$ and ϵ_{abc} is the 3-dimensional totally antisymmetric tensor with $\epsilon_{123} = 1$. In terms of R, how do we represent a general $SO(3)$ transformation on a real 3-vector $\underline{x}$ and what is the geometrical interpretation?

(iii) We define the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy

$$\sigma_a\sigma_b = \delta_{ab}I + i\epsilon_{abc}\sigma_c,$$

I being the 2 by 2 identity matrix. Show that a linear transformation on $\underline{x}\cdot\underline{\sigma} \to \underline{x}'\cdot\underline{\sigma} = A\underline{x}\cdot\underline{\sigma}A^\dagger$ leaves the length of $\underline{x}$ invariant, provided that A is in a particular group G, which you should identify.

(iv) Thus derive R_{ab} in terms of A and the Pauli matrices.

(v) From the expression in (iv), identify a geometrical interpretation when $A = M$.

(vi) What other element of G has the same geometrical interpretation?

4 Define the *commutator* $[g_1,\ g_2]$ of elements g_1, g_2 of a group G.

Consider a transformation on a 4-vector x^μ, $\mu, \nu, \ldots = 0, 1, 2, 3$: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ which leaves $x^\mu \eta_{\mu\nu} x^\nu$ invariant, where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(i) Derive a relation between components of η and those of Λ and hence identify G, the matrix group of the transformations.

(ii) Consider the transformations

$$\Lambda_a{}^\mu{}_\nu = \delta^\mu{}_\nu + \omega_a{}^\mu{}_\nu + \mathcal{O}(\omega_a^2), \qquad a = 1, 2,$$

where $\omega_a{}^\mu{}_\nu$ are infinitesimal. Find the symmetry of $\omega_a{}^{\mu\nu}$ under the exchange of μ and ν.

(iii) Explicitly writing the order of terms you are neglecting, calculate $\Lambda_c = [\Lambda_2,\ \Lambda_1]$ in terms of the components of ω_1 and ω_2 including all calculable terms up to second order.

(iv) Acting on a quantum state, we define the action of the group with the unitary operator $U[\Lambda] = \exp(\omega_{\rho\sigma} M^{\rho\sigma}/2)$, where the anti-hermitian operator $M^{\rho\sigma}$ satisfies $M^{\rho\sigma} = -M^{\sigma\rho}$ and $U[\Lambda]U[\Lambda'] = U[\Lambda\Lambda']$ for $\Lambda, \Lambda' \in G$. Use $U[\Lambda_c]$ to calculate the Lie algebra $L(G)$.

(v) Defining $J_m = \epsilon_{mij} M_{ij}/2$ and $K_i = M_{0i}$, where $i, j, k = 1, 2, 3$, find $[K_i, K_j]$, $[J_i, K_j]$ and $[J_i, J_j]$ in terms of J_k and K_k.

(vi) Rewrite $L(G)$ in terms of $J_j^\pm := (J_j \pm i K_j)/2$.

(vii) Thus identify $L(G)$ as a particular direct sum of simple Lie algebras.

END OF PAPER