UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2021 12:00 pm to 3:00 pm

PAPER 301

QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove Noether's theorem for the theory of a real scalar field $\phi(x)$ with Lagrangian density of the form,

$$\mathcal{L} \;=\; \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi \;-\; V\,(\phi)$$

i) Construct conserved currents corresponding to infinitesimal spacetime translations of the form $x^\mu \to x^\mu - \varepsilon^\mu$ where ε^μ is a constant four vector and use them to define the total energy, E, and 3-momentum $\mathbf{P}$, carried by the field. Using Noether's theorem, show that E and $\mathbf{P}$ are time-independent provided that the fields and their derivatives decay sufficiently fast at spatial infinity.

ii) Find conserved currents corresponding to infinitesimal Lorentz transformations of the form, $x^\mu \to x^\mu - \omega^\mu_{\ \nu} x^\nu$ where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ is a constant antisymmetric tensor. Write down the explicit form of $\omega^{\mu\nu}$ corresponding to an infinitesimal rotation around the z-axis. Hence find an expression for the value of the z-component of angular momentum carried by the field.

iii) Find the most general form of the potential $V(\phi)$ for which the theory has an additional symmetry under which $x^\mu \to \lambda x^\mu$ and,

$$\phi(x) \;\;\to\;\; \lambda^{-1}\,\phi(\lambda^{-1}x)$$

for any $\lambda > 0$ and find the corresponding conserved current.

2 The scalar Yukawa theory describes the interactions of a complex scalar field $\Phi(x)$ of mass M and a real scalar field $\phi(x)$ of mass μ with Lagrangian density,

$$\mathcal{L} \;=\; \partial_\mu\Phi^\star\partial^\mu\Phi - M^2\Phi^\star\Phi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \lambda\phi\Phi^\star\Phi$$

where $\Phi^\star$ is the complex conjugate of Φ and λ is a coupling constant.

Consider the scattering process $\Phi(p) + \phi(k) \to \Phi(p') + \phi(k')$, where p, k, $p' \neq p$ and $k' \neq k$ denote the four-momenta of the initial and final state particles. By applying Dyson's formula and Wick's theorem, evaluate the scattering amplitude for this process to leading non-vanishing order in λ. In your answer you should identify the Wick contractions and the terms in the mode expansion of each field which yield non-zero contributions at this order. You may quote without proof any results from the lectures regarding the canonical quantization of scalar fields that you may need.

3 In this question we consider a Lorentz transformation $x^\mu \to \Lambda^\mu_{\ \nu} x^\nu$ specified by an element,

$$\Lambda \ = \ \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right)$$

of the Lorentz group where,

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} \ = \ \eta^{\rho\mu}\,\eta^{\sigma\nu} - \eta^{\sigma\mu}\,\eta^{\rho\nu}$$

are the Lorentz generators and $\Omega_{\rho\sigma} = -\Omega_{\sigma\rho}$ are real parameters. The Clifford algebra is generated by 4×4 matrices γ^μ, $\mu = 0,1,2,3$ obeying the relations,

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \ = \ 2\eta^{\mu\nu}\mathbb{1}_4$$

where $\mathbb{1}_4$ is the 4×4 unit matrix. The matrices γ^μ can be chosen to obey $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. We also define $\gamma^5 := -i\gamma^0\gamma^1\gamma^2\gamma^3$.

i) Define Lorentz generators $S^{\mu\nu}$ in the spinor representation and show that they obey,

$$[S^{\mu\nu}, \gamma^\rho] \ = \ \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$$

Hence define the spinor representation of the Lorentz group by specifying a 4×4 matrix $S[\Lambda]$ representing Λ. [You do not need to prove that it is a representation]. Show that the hermitian conjugate $S^\dagger[\Lambda]$ of the matrix $S[\Lambda]$ and its inverse $S^{-1}[\Lambda]$ obey,

$$S^\dagger[\Lambda] \ = \ \gamma^0 S^{-1}[\Lambda]\gamma^0$$

ii) Let $\psi(x)$ be a Dirac spinor field. Give the transformation of $\psi(x)$ under Λ and deduce the corresponding transformations of $\partial_\mu\psi(x)$ and of the Dirac adjoint field $\bar{\psi}(x)$.

iii) Write down the Dirac action for a spinor field $\psi(x)$ of mass m and prove that it is invariant under infinitesimal Lorentz transformations defined by expanding Λ to linear order in the parameters $\Omega_{\sigma\rho}$.

iv) The parity transformation P under which $x^\mu = (t, \mathbf{x}) \to (t, -\mathbf{x})$ acts on the Dirac spinor field as,

$$P: \qquad \psi(t,\mathbf{x}) \ \to \ \gamma^0\psi(t,-\mathbf{x})$$

Determine the action of P on each component of $J^\mu_V(t,\mathbf{x}) := \bar{\psi}(x)\gamma^\mu\psi(x)$ and of $J^\mu_A(t,\mathbf{x}) := \bar{\psi}(x)\gamma^5\gamma^\mu\psi(x)$, for $\mu = 0,1,2,3$.

4

a) Starting from an appropriate mode expansion of the Heisenberg picture field operator $\hat{A}_\mu(x)$, which you may state without proof, derive the following integral representation of the photon propagator in Feynman gauge,

$$\langle 0|\mathrm{T}\left[\hat{A}^\mu(x)\hat{A}^\nu(y)\right]|0\rangle \quad = \quad \int_{C_F}\frac{d^4k}{(2\pi)^4}\frac{-i\eta^{\mu\nu}}{k^2}\exp\left[-ik\cdot(x-y)\right]$$

where you should specify, with appropriate justification, the contour C_F.

b) A complex scalar field $\Phi(x)$ of mass M has Lagrangian density,

$$\mathcal{L} \quad = \quad \partial_\mu\Phi^\star\partial^\mu\Phi - M^2\Phi^\star\Phi$$

where $\Phi^\star$ is the complex conjugate of Φ. Identify a $U(1)$ global symmetry of this theory and explain what it means to *gauge* this symmetry. By defining an appropriate covariant derivative, write down a gauge-invariant Lagrangian density describing the complex scalar field $\Phi(x)$ coupled to an electromagnetic field with four-vector potential $A_\mu(x)$ and gauge coupling e.

Identify the interaction terms in the Lagrangian and, by considering how each of these terms contributes to scattering amplitudes, write down the momentum-space Feynman rules for this theory. You should specify the Feynman rules associated to internal lines and vertices appearing in Feynman diagrams. You are not required to discuss additional factors corresponding to external lines.

END OF PAPER