UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 16 June, 2021 12:00 pm to 2:00 pm

PAPER 220

RANDOM PLANAR GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of a compact $\mathbb{R}$-*tree* $(\mathcal{T}, d)$ and the *multiplicity* of $a \in \mathcal{T}$.

(b) Suppose that $g \colon [0,1] \to \mathbb{R}_+$ is a continuous function with $g(0) = g(1) = 0$. Give the definition of the metric space $(\mathcal{T}_g, d_g)$ encoded by g. *[You do not need to prove that* $(\mathcal{T}_g, d_g)$ *is a compact* $\mathbb{R}$*-tree and may assume that this is the case for the remainder of the question.]*

(c) Prove or disprove: if $(\mathcal{T}, d)$ is a compact $\mathbb{R}$-tree then every $a \in \mathcal{T}$ has finite multiplicity.

(d) Suppose that $f, g \colon [0,1] \to \mathbb{R}_+$ are continuous functions with $f(0) = f(1) = g(0) = g(1) = 0$. Prove or disprove: if $(\mathcal{T}_f, d_f) = (\mathcal{T}_g, d_g)$ then $f = g$.

(e) Give the definition of the *Hausdorff* and the *Gromov-Hausdorff metrics*. Prove or disprove: the set of compact $\mathbb{R}$-trees is a compact subset with respect to the Gromov-Hausdorff topology.

(f) Suppose that $(\mathcal{T}, d)$ is a compact $\mathbb{R}$-tree. Show that there exists a continuous function $g \colon [0,1] \to \mathbb{R}_+$ with $g(0) = g(1) = 0$ so that $(\mathcal{T}_g, d_g) = (\mathcal{T}, d)$. *[You may assume without proof that the map which associates* g *to* $(\mathcal{T}_g, d_g)$ *is a continuous map from the space of continuous functions* $[0,1] \to \mathbb{R}_+$ *which take the value* 0 *at* $t = 0, 1$ *with the metric* $\|f - g\|_\infty := \sup_{t \in [0,1]} |f(t) - g(t)|$ *to the space of compact metric spaces with the Gromov-Hausdorff metric.]*

2

(a) Fix $\alpha \in (0,1)$ and suppose that $g\colon [0,1] \to \mathbb{R}_+$ is an α-Hölder continuous function with $g(0) = g(1) = 0$.

(i) For each $0 \leqslant s,t \leqslant 1$ let $m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r)$. Show that for every $0 \leqslant s_1, \ldots, s_n \leqslant 1$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ we have that

$$\sum_{i,j} \lambda_i \lambda_j m_g(s_i, s_j) \geqslant 0.$$

(ii) Give the definition of the *Brownian snake* driven by g and show that for each $\epsilon > 0$ it has an $(\alpha/2 - \epsilon)$-Hölder continuous modification.

(b) Give the definition of the set $\mathcal{Q}_n$ of *rooted quadrangulations*, and the set $\mathcal{Q}_n^\bullet$ of *rooted and pointed quadrangulations*. Explain why $\#\mathcal{Q}_n = \#\mathcal{M}_n$ where $\mathcal{M}_n$ is the set of rooted planar maps with n edges by describing the *trivial bijection. [You do not need to prove that the trivial bijection is a bijection.]*

(c) Suppose that S_k is a simple symmetric random walk starting from 0. Define the process V_k as follows. We set $V_0 = 0$. Given that $V_0, \ldots, V_k$ have been defined, we let $V_{k+1} = V_k + \xi_k$ if $S_{k+1} - S_k = 1$ where $\mathbb{P}[\xi_k = 1] = \mathbb{P}[\xi_k = 0] = \mathbb{P}[\xi_k = -1] = 1/3$ and ξ_k is independent of $V_0, \ldots, V_k$ and all of S. If $S_{k+1} - S_k = -1$, then we set $V_{k+1} = 0$ if $S_{k+1} < \min\{S_j : 0 \leqslant j \leqslant k\}$ and otherwise $V_{k+1} = V_\zeta$ where $\zeta = \max\{j \leqslant k : S_j = S_{k+1}\}$.

(i) For $m \in \mathbb{Z}$ and $m \leqslant 0$, let G_m be the number of k so that $V_k = m$ when $m = \min\{V_j : 0 \leqslant j \leqslant k\}$. Show that $\mathbb{P}[G_m \geqslant j] = (1 - 1/6)^{j-1}$ for each $j \geqslant 1$.

(ii) Suppose that (q, e, v_*) is a uniformly random element of $\mathcal{Q}_n^\bullet$ where e is the root edge and v_* is the distinguished vertex. Show that there exists a constant $C > 0$ so that $\mathbb{P}[\deg(v_*) \leqslant C \log n] \to 1$ as $n \to \infty$. *[You may use without proof that if $\sigma = \min\{k \geqslant 0 : S_k = -1\}$ then $\mathbb{P}[\sigma = 2n+1] \geqslant cn^{-3/2}$ where $c > 0$ is a constant.]*

3

(a) Give the definition of a *compact* $\mathbb{H}$*-hull* and its *half-plane capacity* (hcap).

(b) Suppose that A, C are compact $\mathbb{H}$-hulls. Prove that $\mathrm{hcap}(A) = \lim_{y\to\infty} y\mathbb{E}_{iy}[\mathrm{Im}(B(\tau))]$ where B is a complex Brownian motion. Prove also the following.

(i) If $A \subseteq C$ then $\mathrm{hcap}(A) \leqslant \mathrm{hcap}(C)$.

(ii) $\mathrm{hcap}(A \cup C) \leqslant \mathrm{hcap}(A) + \mathrm{hcap}(C)$.

(c) Suppose that (A_n) is a sequence of compact $\mathbb{H}$-hulls. Prove or disprove the following statements. *[You may use results about* hcap *stated in lectures provided you state them clearly.]*

(i) If $\mathrm{diam}(A_n) \to \infty$ then $\mathrm{hcap}(A_n) \to \infty$.

(ii) If $\mathrm{hcap}(A_n) \to \infty$ then $\mathrm{diam}(A_n) \to \infty$.

(d) Suppose that γ is an SLE_κ in $\mathbb{H}$ from 0 to ∞. *[You may use results about Bessel processes from lectures provided you state them clearly.]*

(i) Prove that γ almost surely intersects $\partial\mathbb{H} \setminus \{0\}$ if and only if $\kappa > 4$.

(ii) Prove that γ almost surely intersects $\partial\mathbb{H}$ infinitely many times if and only if $\kappa > 4$.

(iii) Prove that the set $\{\gamma(t) : t \in \mathbb{R}_+,\ \gamma(t) \in \partial\mathbb{H}\}$ almost surely has zero Lebesgue measure if $\kappa \in (0, 8)$. *[You may assume that if $\kappa \in (4, 8)$ then $\mathbb{P}[\tau_x = \tau_y] > 0$ for all $0 < x < y$ where for each $x \in \mathbb{R}$ we let τ_x be the first time that γ disconnects x from ∞.]*

4

(a) State the *conformal Markov property*. Show that if (g_t) is a Loewner evolution driven by U_t which satisfies the conformal Markov property then there exists $\kappa \geqslant 0$ and a Brownian motion B_t so that $U_t = \sqrt{\kappa} B_t$.

(b) State the *locality property* for SLE_6.

(c) Prove that SLE_6 satisfies the locality property. *[You may assume the following facts. Suppose that (A_t) is a non-decreasing family of compact $\mathbb{H}$-hulls which are locally growing, parameterized by half-plane capacity with $A_0 = 0$ and with Loewner driving function U and Loewner evolution (g_t). Let $D \subseteq \mathbb{H}$ be a simply connected domain with 0 on its boundary and let $\psi \colon D \to \mathbb{H}$ be a conformal transformation with $\psi(0) = 0$. Assume that $T > 0$ is such that $A_T \subseteq D$, let $\widetilde{A}_t = \psi(A_t)$, and $\widetilde{g}_t = g_{\widetilde{A}_t}$ for $t \in [0, T]$. Then the maps $(\widetilde{g}_t)$ satisfy $\partial_t \widetilde{g}_t(z) = \partial_t \widetilde{a}(t)/(\widetilde{g}_t(z) - \widetilde{U}_t)$, $\widetilde{g}_0(z) = z$, where $\widetilde{U}_t = \psi_t(U_t)$, $\psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1}$ and $\widetilde{a}(t) = \int_0^t 2(\psi_s'(U_s))^2 ds$ for $t \in [0, T]$. You may also assume without proof the formula $\partial_t \psi_t(U_t) = -3\psi_t''(U_t)$.]*

(d) Suppose that γ_1 is an SLE_6 in $\mathbb{H}$ from -1 to ∞ and $\tau_1 = \inf\{t \geqslant 0 : \gamma_1(t) \notin B(-1, 1)\}$. Given $\gamma_1|_{[0,\tau_1]}$, let γ_2 be an SLE_6 in the unbounded component of $\mathbb{H} \setminus \gamma_1([0, \tau_1])$ from 1 to ∞ and let $\tau_2 = \inf\{t \geqslant 0 : \gamma_2(t) \notin B(1, 1)\}$. Show that given $\gamma_2|_{[0,\tau_2]}$, the conditional law of $\gamma_1|_{[0,\tau_1]}$ is that of an SLE_6 in the unbounded component of $\mathbb{H} \setminus \gamma_2([0, \tau_2])$ from -1 to ∞ stopped upon leaving $B(-1, 1)$.

(e) Suppose that $x > 0$ and γ is an SLE_6 in $\mathbb{H}$ from 0 to x. Let τ_x be the first time that γ disconnects x from ∞. Show that $\gamma|_{[0,\tau_x]}$ has the law of an SLE_6 in $\mathbb{H}$ from 0 to ∞ stopped at the first time that it disconnects x from ∞.

END OF PAPER