UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2021 12:00 pm to 2:00 pm

PAPER 204

PERCOLATION AND RELATED TOPICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove the subadditive-inequality theorem, namely the following. Let $(a_n : n \geqslant 1)$ be a sequence of reals satisfying the subadditive inequality $a_{r+s} \leqslant a_r + a_s$ for $r, s \geqslant 1$. Show the existence of the limit

$$\lambda = \lim_{n \to \infty} a_n/n,$$

and prove that $\lambda = \inf\{a_r/r : r \geqslant 1\}$.

Let b_n be the number of n-step self-avoiding walks on the square lattice $\mathbb{Z}^2$ starting at the origin.

(a) Show that the limit $\kappa = \lim_{n\to\infty} b_n^{1/n}$ exists.

(b) Show that $2 \leqslant \kappa \leqslant 3$.

(c) By considering walks whose horizontal steps are always rightwards, or otherwise, prove that $\kappa > 2$.

2

(a) Explain what is meant by the site percolation model on the three-dimensional cubic lattice with vertex set $\mathbb{Z}^3$ and parameter p. Write $\mathbb{P}_p$ for the corresponding probability measure.

(b) Define the *percolation probability* $\theta(p)$ and the *critical probability* p_{c}.

(c) Let Λ_n be the set of vertices lying within the box $[-n, n]^3$. Write $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$, and let A_n be the event that there exists an open path from the origin 0 to $\partial\Lambda_n$. Show that $\mathbb{P}_p(A_n) \downarrow \theta(p)$ as $n \to \infty$. Deduce that the function θ is *right continuous* in that $\lim_{p'\downarrow p} \theta(p') = \theta(p)$.

(d) Explain the construction of the monotone coupling $(\eta_p : p \in [0, 1])$ of all site percolation models on $\mathbb{Z}^3$ in terms of independent, uniformly distributed random variables.

(e) Let I_p be the event that η_p contains an infinite open path starting at 0. Show that

$$\theta(p) - \lim_{p'\uparrow p} \theta(p') = \mathbb{P}\big(I_p \cap \{M = p\}\big),$$

where $M = \inf\{p : I_p \text{ occurs}\}$ and $\mathbb{P}$ is the appropriate probability measure.

(f) Deduce that θ is *continuous* on the half-open interval $(p_{\mathrm{c}}, 1]$.

[*Any general result may be used without proof, but should be stated carefully.*]

3 Let $G = (V, E)$ be a finite, connected graph, and let $\Omega = \{0,1\}^E$. Define the random-cluster measure $\phi_{p,q}$ with parameters p and q on the sample space Ω. For $\omega \in \Omega$, write $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ for the set of 'open' edges in ω, and $k(\omega)$ for the number of components of the 'open graph' $(V, \eta(\omega))$.

State the FKG (positive-association) theorem for $\phi_{p,q}$, taking care to include the appropriate conditions on p and q. By considering the graph with two vertices joined by two parallel edges, or otherwise, give a counterexample to the FKG inequality when $p, q \in (0,1)$. [You may use the usual definition of the random-cluster model on this two-vertex graph.]

Let $p = \frac{1}{2}$. Show that, in the limit $q \downarrow 0$, $\phi_{p,q}$ converges to the uniform probability measure on the set of connected subgraphs of G. That is,

$$\phi_{p,q}(\omega) \to \begin{cases} \dfrac{1}{|\mathcal{C}|} & \text{if } \omega \in \mathcal{C}, \\ 0 & \text{if } \omega \notin \mathcal{C}, \end{cases} \tag{*}$$

where $\mathcal{C} \subseteq \Omega$ is the set of configurations ω such that the subgraph $(V, \eta(\omega))$ of G is connected. It may be helpful to write

$$\phi_{p,q}(\omega) = \frac{1}{Z} f(\omega) \qquad \text{where } f(\omega) = \left(\frac{p}{1-p}\right)^{|\eta(\omega)|} q^{k(\omega)}, \ Z = \sum_{\omega'} f(\omega').$$

Suppose now that $p, q \to 0$ in such a way that $q/p \to 0$. State and prove the limit of $\phi_{p,q}(\omega)$, as in (*) above but with $\mathcal{C}$ replaced by a certain set $\mathcal{T}$ to be specified. You may use without proof the fact that $|\eta(\omega)| + k(\omega) \geqslant |V|$, with equality if and only if the open graph contains no circuits.

4 Define the *contact model* $\xi = (\xi_t : t \geqslant 0)$ on the infinite line $\mathbb{Z}$ with infection rate $\lambda > 0$ and cure rate $\mu > 0$. Describe the *graphical representation* of the contact model in terms of independent Poisson processes in the space $\mathbb{Z} \times [0, \infty)$, and illustrate your answer with a diagram. Write $\mathbb{P}_{\lambda,\mu}$ for the corresponding probability measure.

Let $\xi^A = (\xi_t^A : t \geqslant 0)$ be defined by the above graphical representation, where A denotes the initial set of infectives (at time 0). Show the following.

(a) The process ξ is *increasing* in that $\xi_t^A \subseteq \xi_t^B$ if $A \subseteq B$.

(b) Moreover, ξ is *additive* in that $\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B$.

(c) The *duality relation* holds, namely

$$\mathbb{P}_{\lambda,\mu}(\xi_t^A \cap B \neq \varnothing) = \mathbb{P}_{\lambda,\mu}(\xi_t^B \cap A \neq \varnothing)$$

for $A, B \subseteq \mathbb{Z}$.

Define the *survival probability* $\theta(\lambda, \mu)$ and the *critical point* $\lambda_{\mathrm{c}}(\mu)$. Prove that

$$\theta(\lambda, \mu) = \lim_{t \to \infty} \mathbb{P}_{\lambda,\mu}(\xi_t^{\mathbb{Z}}(x) = 1),$$

for all $x \in \mathbb{Z}$.

END OF PAPER