UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 17 June, 2021 12:00 pm to 3:00 pm

PAPER 202

STOCHASTIC CALCULUS AND APPLICATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***FOUR*** *questions.*
There are ***SIX*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be of finite variation and right-continuous.

(a) Show that there exist non-decreasing right-continuous functions g and h such that $f = g - h$.

(b) Suppose $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is continuous. Show that

$$\sum_{k=1}^{2^n} \alpha(t_{k-1}^{(n)})(f(t_k^{(n)}) - f(t_{k-1}^{(n)})) \to \int_0^1 \alpha(s) df(s) \text{ as } n \to \infty$$

where $t_k^{(n)} = k2^{-n}$.

(c) Suppose that f is continuous. Show that

$$2\int_0^t f(s) df(s) = f(t)^2.$$

2 Let X be a continuous process adapted to a filtration satisfying the usual conditions such that $X_0 = 0$. For each $n \geqslant 1$ and $t \geqslant 0$, let

$$[X]_t^{(n)} = \sum_{k=1}^{\infty} (X_{t_k^{(n)} \wedge t} - X_{t_{k-1}^{(n)} \wedge t})^2,$$

where $t_k^{(n)} = k2^{-n}$.

(a) Assuming that X is a uniformly bounded martingale, show that there exists an adapted process $[X]$ such that

(i) $\mathbb{E}(\sup_{t \geqslant 0} |[X]_t^{(n)} - [X]_t|^2) \to 0$.

(ii) $X^2 - [X]$ is an L^2 bounded martingale.

(iii) $[X]$ is non-decreasing.

[You may use the following fact without proof:

Fact. For $n \geqslant 1$ and $t \geqslant 0$, let

$$M_t^{(n)} = \sum_{k=1}^{\infty} X_{t_{k-1}^{(n)}} (X_{t_k^{(n)} \wedge t} - X_{t_{k-1}^{(n)} \wedge t}).$$

Then $M^{(n)}$ is a continuous martingale for each n and there exists a continuous martingale M such that $\mathbb{E}(\sup_{t \geqslant 0} M_t^2) < \infty$ and $\mathbb{E}(\sup_{t \geqslant 0} |M_t^{(n)} - M_t|^2) \to 0$ as $n \to \infty$.]

(b) Now assume that X is a local martingale. Show that there exists an adapted process $[X]$ such that

(i) $\sup_{0 \leqslant t \leqslant u} |[X]_t^{(n)} - [X]_t| \to 0$ in probability for all $u \geqslant 0$.

(ii) $X^2 - [X]$ is a local martingale.

(c) Show that the process $[X]$ in part (b) has the property that $\sup_{t \geqslant 0} \mathbb{E}([X]_t) < \infty$ if and only if $\mathbb{E}(\sup_{t \geqslant 0} X_t^2) < \infty$.

3 (a) Let M be a positive continuous martingale, such that $\log M$ is bounded. Show that

$$\mathbb{E}(M_1 \log M_1) = \mathbb{E}(M_0 \log M_0) + \frac{1}{2}\mathbb{E}\left(\int_0^1 \frac{d[M]_t}{M_t}\right)$$

where $[M]$ is the quadratic variation of M.

For the rest of the problem, let B be a standard Brownian motion, let f be a smooth function taking values in $[a, b]$ where $0 < a < b < \infty$, and assume the derivative f' is bounded. For $t \in [0, 1]$ and $x \in \mathbb{R}$, let

$$U(t, x) = \mathbb{E}\{f(x + B_{1-t})^2\}.$$

(b) Let $M_t = U(t, W_t)$ where W is a Brownian motion independent of B. By directly computing conditional expectations and the definition of Brownian motion, show that M is a martingale with respect to the filtration generated by W.

(c) Establish the identity

$$\mathbb{E}\{f(W_1)^2 \log\left(f(W_1)^2\right)\} = \mathbb{E}\{f(W_1)^2\} \log \mathbb{E}\{f(W_1)^2\} + \frac{1}{2}\mathbb{E}\left\{\int_0^1 \frac{\left(\frac{\partial U}{\partial x}(t, W_t)\right)^2}{U(t, W_t)} dt\right\}.$$

[Hint: $f(W_1)^2 = U(1, W_1)$ and $\mathbb{E}\{f(W_1)^2\} = U(0, 0)$. You may assume that U is smooth.]

(d) Prove that

$$\mathbb{E}\{f(W_1)^2 \log\left(f(W_1)^2\right)\} \leqslant \mathbb{E}\{f(W_1)^2\} \log \mathbb{E}\{f(W_1)^2\} + 2\mathbb{E}\{f'(W_1)^2\}.$$

[You may use the fact that $\frac{\partial U}{\partial x}(t, x) = 2\,\mathbb{E}\{f(x + B_{1-t}) f'(x + B_{1-t})\}$ without justification.]

4 (a) Let X be a continuous, non-negative local martingale such that $X_0 = 1$ and $X_t \to 0$ almost surely as $t \to \infty$. For each $a > 1$, let $\tau_a = \inf\{t \geqslant 0 : X_t > a\}$. Show that

$$\mathbb{P}(\tau_a < \infty) = \mathbb{P}(\sup_{t \geqslant 0} X_t > a) = 1/a.$$

[Hint: compute the expected value of $X_{t\wedge\tau_a} = a\mathbf{1}_{\{\tau_a \leqslant t\}} + X_t \mathbf{1}_{\{\tau_a > t\}}$.]

Let M be a continuous local martingale with $M_0 = 0$ and $[M]_\infty = \infty$ almost surely.

(b) State the Dambis–Dubins–Schwarz theorem in terms of M. Give a proof of this theorem under the additional assumption that the quadratic variation process $[M]$ is strictly increasing almost surely.

(c) Show that $M_t - \frac{1}{2}[M]_t \to -\infty$ almost surely as $t \to \infty$. [Hint: You may use without proof the fact that $\frac{W_t}{t} \to 0$ almost surely as $t \to \infty$, where W is a Brownian motion.]

(d) Show that

$$\mathbb{P}(\sup_{t \geqslant 0}\{M_t - \tfrac{1}{2}[M]_t\} > y) = e^{-y}$$

for all $y > 0$. [Hint: Consider the local martingale $X = e^{M - \frac{1}{2}[M]}$.]

5 Consider the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \qquad (*)$$

where W is a scalar Brownian motion and $b, \sigma : \mathbb{R} \to \mathbb{R}$ are given functions.

(a) In the context of the stochastic differential equation $(*)$, define the following terms:

(i) weak solution;

(ii) strong solution;

(iii) uniqueness in law; and,

(iv) pathwise uniqueness.

For the rest of the question, specialise to the case where $\sigma(x) = 1$ for all x and b is continuous and bounded.

(b) Let X be a solution of $(*)$. Show that there exists a strictly increasing function g such that $Y = g(X)$ is a local martingale.

(c) Verify that the function g found in part (b) is such that the function $h = g' \circ g^{-1}$ is Lipschitz. Hence, prove that equation $(*)$ has a pathwise unique strong solution. [You may use any standard results on the existence and uniqueness of the solutions of stochastic differential equations as long as they are carefully stated.]

6 Suppose that a market has $1+d$ assets with prices given by

$$dS_t^{(0)} = S_t^{(0)} r_t dt$$
$$dS_t^{(i)} = S_t^{(i)} \left(\mu_t^{(i)} \ dt + \sum_{k=1}^{d} \sigma_t^{(i,k)} dW_t^{(k)} \right)$$

for $1 \leqslant i \leqslant d$ where the adapted processes $r, \mu^{(i)}, \sigma^{(i,k)}$ are bounded and continuous, and where the $W^{(k)}$ are Brownian motions.

(a) Given an investor's initial wealth $x \geqslant 0$, what is an x-admissible trading strategy?

(b) What is an arbitrage?

(c) Suppose that the matrix $\sigma_t(\omega)$ is invertible for all (t, ω) and that the process σ^{-1} is bounded. Show that the market has no arbitrage. Standard results from stochastic calculus may be used without proof, but they must be stated clearly. If you use a fundamental theorem of asset pricing, you must prove it.

(d) A market is said to satisfy the Law of One Price if it has the property that $S_T^{(i)} = S_T^{(j)}$ almost surely for some non-random $T > 0$ implies $S_t^{(i)} = S_t^{(j)}$ almost surely for all $0 \leqslant t \leqslant T$. Give an example of a market model with no arbitrage which does *not* obey the Law of One Price. [Hint: You may use the fact that any positive solution of the stochastic differential equation $dX_t = X_t^2 dW_t$ is a strictly local martingale.]

END OF PAPER