UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 8 June, 2021 12:00 pm to 3:00 pm

PAPER 201

ADVANCED PROBABILITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***FOUR*** *questions.*
There are ***SIX*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, all notions are to be understood with reference to a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) What is meant by the conditional expectation $\mathbb{E}(X|\mathcal{G})$ of an integrable random variable X given a sub-σ-algebra $\mathcal{G}$?

(b) Show that the conditional expectation is unique, in a sense which you should make precise.

(c) Show that, if $\mathcal{H}$ is another sub-σ-algebra, which is independent of $\mathcal{G}$, then

$$\mathbb{E}(X|\mathcal{G} \cap \mathcal{H}) = \mathbb{E}(X) \quad \text{almost surely.}$$

(d) Let X_1, X_2 be independent $N(0,1)$ random variables and let $\mathcal{G}$ be the σ-algebra generated by the random variable $X_1 + X_2$. Find $\mathbb{E}(X_1|\mathcal{G})$.

(e) Let X be an integrable random variable and let $\mathcal{G}, \mathcal{H}$ be sub-σ-algebras. Consider the equation

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{G} \cap \mathcal{H}) \quad \text{almost surely.}$$

Show the validity of this equation in the following cases: (i) $\mathcal{G} \subseteq \mathcal{H}$, (ii) $\mathcal{H} \subseteq \mathcal{G}$, (iii) $\mathcal{G}$ and $\mathcal{H}$ are independent.

(f) Is the equation in part (e) true in general? Justify your answer.

2 Let $(X_n)_{n \geqslant 0}$ be a non-negative supermartingale.

(a) Show that, for all $a, b \geqslant 0$ with $a < b$, the number of upcrossings $U[a,b]$ of the interval $[a,b]$ by $(X_n)_{n \geqslant 0}$ satisfies

$$\mathbb{E}(U[a,b]) \leqslant a/(b-a).$$

(b) Deduce that $(X_n)_{n \geqslant 0}$ converges almost surely as $n \to \infty$.

[You may assume the optional stopping theorem. If you use any other result of martingale theory you should prove it.]

3 Let $(S_n)_{n\geqslant 0}$ be a simple discrete-time random walk on the integers, starting from 0, with

$$\mathbb{P}(S_1 = 1) = p, \quad \mathbb{P}(S_1 = -1) = 1 - p, \quad p \in (0,1).$$

Write $(S_t)_{t\geqslant 0}$ for the continuous-time process obtained by linear interpolation of the discrete-time walk between integer times. Thus $S_{n+t} = (1-t)S_n + tS_{n+1}$ for all non-negative integers n and all $t \in [0,1]$.

(a) Show that $(S_n - \mu n)_{n\geqslant 0}$ is a discrete time martingale for some μ, to be determined.

(b) Compute the variance of S_n and hence show that the rescaled process $S_t^{(n)} = n^{-1}S_{nt}$ satisfies

$$\mathbb{E}\left(\sup_{t\in[0,1]}\left|S_t^{(n)} - \mu t\right|^2\right) \leqslant \frac{4}{n}.$$

(c) Show that, for all $\theta \in \mathbb{R}$, for some $\psi(\theta) \in \mathbb{R}$ to be determined, the following process is a martingale

$$Z_n = e^{\theta S_n - \psi(\theta)n}.$$

(d) Fix $\varepsilon > 0$ and n and consider the event

$$A = \{S_k - \mu k \geqslant \varepsilon n \text{ for some } k \leqslant n\}.$$

For $\theta > 0$, by considering the value of Z_T on A for a suitable stopping time T, show that

$$\mathbb{P}(A) \leqslant e^{-(\theta(\mu+\varepsilon)-\psi(\theta))n}$$

and deduce that

$$\mathbb{P}\left(\sup_{t\in[0,1]}|S_t^{(n)} - \mu t| \geqslant \varepsilon\right) \leqslant e^{-n\psi^*(\mu+\varepsilon)} + e^{-n\psi^*(\mu-\varepsilon)}$$

where ψ^* is the Legendre transform of ψ.

4

(a) What does it mean to say that a random process $(X_t)_{t \geqslant 0}$ is a Brownian motion in $\mathbb{R}^d$?

(b) Show that, if $(X_t)_{t \geqslant 0}$ is a Brownian motion in $\mathbb{R}^d$ and if U is an orthogonal $d \times d$ matrix, then the process $(UX_t)_{t \geqslant 0}$ is also a Brownian motion in $\mathbb{R}^d$.

Fix $a > 0$ and let $(A_t^-)_{t \geqslant 0}$, $(A_t^+)_{t \geqslant 0}$ and $(B_t)_{t \geqslant 0}$ be independent one-dimensional Brownian motions, starting from $-a$, a and 0 respectively. Set

$$T = \inf\{t \geqslant 0 : A_t^- = A_t^+\}$$

and define on the event $\{T \leqslant t\}$

$$Z_t = A_T^+ + (B_t - B_T).$$

(c) By considering an orthogonal transformation of $(A_t^-, A_t^+)_{t \geqslant 0}$, or otherwise, show that $T < \infty$ almost surely and find a density function for T.

(d) For $s, t > 0$ with $s \leqslant t$, show that

$$\mathbb{P}(T \leqslant s \text{ and } Z_t \leqslant z) = \int_0^s \Phi\left(\frac{z}{\sqrt{t - u/2}}\right) \frac{a}{\sqrt{\pi u^3}} e^{-a^2/u} du$$

where Φ denotes the standard normal distribution function.

5

(a) State and prove the Skorokhod embedding theorem for random walks with steps of mean 0 and variance $\sigma^2 < \infty$.

(b) Deduce the central limit theorem. [You may use any form of the law of large numbers, without proof. If you use Donsker's invariance principle, you should prove it.]

6 In all parts of this question, it is to be assumed that we consider the case of a real-valued random process.

(a) What does it mean to say that $(X_t)_{t \geqslant 0}$ is a Lévy process?

(b) State the Lévy–Khinchin theorem.

(c) State how a Lévy process with Lévy triple (a, b, K) can be constructed from a Brownian motion and a suitable Poisson random measure.

(d) In each of the following cases, find necessary and sufficient conditions on the Lévy triple (a, b, K) such that the associated Lévy process $(X_t)_{t \geqslant 0}$ has the given property:

(i) $(X_t(\omega))_{t \geqslant 0}$ is differentiable for almost all $\omega \in \Omega$,

(ii) $(X_t(\omega))_{t \geqslant 0}$ is continuous for almost all $\omega \in \Omega$,

(iii) $(X_t)_{t \geqslant 0}$ is an integrable process,

(iv) $(X_t^2)_{t \geqslant 0}$ is an integrable process.

[Justify your answers. You may use without proof any property of Brownian motion or Poisson random measures, provided that you state it clearly.]

END OF PAPER