UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 14 June, 2021 12:00 pm to 2:00 pm

PAPER 161

TOPICS IN COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let G be a graph with n vertices and m edges, with $m \geqslant 6n$. Prove that if G is drawn in the plane, then there must be at least cm^3/n^2 crossings, where $c > 0$ is an absolute constant. [You may assume Euler's formula.]

(ii) Let $P_1, \ldots, P_m$ be distinct real polynomials in one variable of degree d, let $x_1, \ldots, x_n$ be distinct real numbers, and let $y_1, \ldots, y_n$ be real numbers. By modifying the proof of the Szemerédi-Trotter theorem appropriately, prove that the number of pairs (i, j) such that $P_i(x_j) = y_j$ is at most $C(m + n + d^{1/3}m^{2/3}n^{2/3})$, where C is an absolute constant.

2 (i) Let X and Y be random variables that each take finitely many possible values. Prove that $H(X,Y) \leqslant H(X) + H(Y)$. Deduce that if $X_1, \ldots, X_n$ are such random variables, then $H(X_1, \ldots, X_n) \leqslant H(X_1) + \cdots + H(X_n)$. [You may use either the formula or the abstract approach.]

(ii) State and prove Shearer's lemma.

(iii) Let $\mathcal{A}$ and $\mathcal{F}$ be two collections of subsets of $[n]$. Suppose that for every $i \in [n]$ there are at least t sets $F \in \mathcal{F}$ such that $i \in F$. For each $F \in \mathcal{F}$, define the *trace* $T_F(\mathcal{A})$ to be $\{A \cap F : A \in \mathcal{A}\}$. Deduce from Shearer's lemma that

$$|\mathcal{A}| \leqslant \Big(\prod_{F \in \mathcal{F}} |T_F(\mathcal{A})| \Big)^{1/t}.$$

[Hint: Let A be an element of $\mathcal{A}$ chosen uniformly at random and let $X_i = 1$ if $i \in A$ and 0 otherwise.]

3 (i) State and prove Alon's combinatorial Nullstellensatz.

(ii) Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Suppose that the permanent of A is non-zero. Let $b \in \mathbb{F}^n$ and let $S_1, \ldots, S_n$ be subsets of $\mathbb{F}$ of size 2. Prove that there exists $x \in S_1 \times \cdots \times S_n$ such that the vectors Ax and b differ in all n coordinates.

[Recall that the permanent is defined using the formula for the determinant but without the signs.]

4 (i) Let p be a prime, let $\mathcal{A}$ be a collection of subsets of $[n]$, and let $E \subset \mathbb{F}_p$ of size m. Suppose that $|A| \notin E$ for every $A \in \mathcal{A}$ and that $|A \cap B| \in E$ for every pair of distinct elements A, B of $\mathcal{A}$. Show that

$$|\mathcal{A}| \leqslant \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}.$$

(ii) Let p be a prime, let $n = p^3$, and define a graph whose vertices are the subsets of $[n]$ of size p^2, with A joined to B if and only if $|A \cap B|$ is a multiple of p. Using the result in (i), prove that the largest independent set in the graph has size at most

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p-1}.$$

Show that this is at most p^{3p}.

(iii) Let q be a prime between p^2 and $2p^2$. Using the result in (i) again (but with p replaced by q) prove that the largest clique in the graph also has size at most p^{3p}.

(iv) Deduce that if $k = p^{3p}$, then the Ramsey number $R(k,k)$ is at least $\binom{p^3}{p^2}$. Show that this is larger than any polynomial function of k.

END OF PAPER