UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 24 June, 2021 12:00 pm to 3:00 pm

PAPER 160

REPRESENTATION THEORY OF SYMMETRIC GROUPS

Before you begin please read these instructions carefully.

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

All representations on this exam are assumed to be finite-dimensional.
Unless otherwise stated, they are over the field $\mathbb{C}$.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, let $\mathbb{F}$ be an arbitrary field.

For a partition μ, the μ-Young permutation module over $\mathbb{F}$ is denoted by M^μ, and the μ-Specht module over $\mathbb{F}$ by $\mathcal{S}^\mu$. Let n be a natural number and λ be a partition of n.

(a) Let v and w be λ-tableaux such that $\mathfrak{b}_v \cdot \{w\} \neq 0$. Show that there exists $h \in C(v)$ such that $h \cdot \{v\} = \{w\}$.

(b) State and prove James's submodule theorem.

(c) Fix a λ-tableau t and consider the λ'-tableau t', the transpose of t, obtained from t by interchanging the rows and columns. Fix a (1^n)-tableau u.

(i) Show that

$$\begin{aligned} \theta \ : \quad M^{\lambda'} &\longrightarrow \mathcal{S}^\lambda \otimes \mathcal{S}^{(1^n)} \\ \{g \cdot t'\} &\longmapsto g \cdot \big(e(t) \otimes e(u)\big) \end{aligned}$$

for all $g \in S_n$, extended $\mathbb{F}$–linearly, is a well-defined and surjective $\mathbb{F}S_n$–homomorphism.

(ii) Writing $\theta\big(e(t')\big) = m \otimes e(u)$ for some $m \in \mathcal{S}^\lambda$, show that $\langle m, \{t\}\rangle = |R(t)|$.

(iii) Now suppose $\mathrm{char}(\mathbb{F}) = 0$. Show that $\ker(\theta) = (\mathcal{S}^{\lambda'})^\perp$.

Hence write down an $\mathbb{F}S_n$–isomorphism from $\mathcal{S}^\lambda \otimes \mathcal{S}^{(1^n)}$ to $(\mathcal{S}^{\lambda'})^*$: you should give the image of $e(w) \otimes e(u)$ for every λ-tableau w.

[As usual, V^* denotes the dual of V. If V is an $\mathbb{F}S_n$–module then the S_n–action on $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ is given by $(g \cdot \phi)(v) := \phi(g^{-1} \cdot v)$ for $g \in S_n$, $\phi \in V^*$ and $v \in V$.]

2 For a partition μ, the μ-Specht module over $\mathbb{C}$ is denoted by $\mathcal{S}^\mu$ and its character by χ^μ. In the usual notation from lectures, $\psi^\lambda = \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \cdot \xi^{\lambda - \mathrm{id} + \pi}$ for integer compositions λ.

(a) (i) Let λ be an integer composition and i be a natural number. Show that $\psi^\mu = -\psi^\lambda$, where

$$\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \ldots).$$

(ii) State and prove the restriction version of the Branching Rule.
[You may assume that $\psi^\lambda = \chi^\lambda$ whenever λ is a partition. You may also use the following result without proof: if λ is an integer composition of $n = m + k$, then $\xi^\lambda\downarrow_{S_m \times S_k} = \sum_{\mu \vDash k} \xi^{\lambda - \mu} \# \xi^\mu$.]

In parts (b)–(d) below, you may use general results from the course without proof, provided they are stated clearly.

(b) Decompose the following $\mathbb{C}S_4$–module V into a direct sum of irreducible modules:

$$V = \left(\mathcal{S}^{(2)}\uparrow_{S_2}^{S_4}\right) \otimes \left(\mathcal{S}^{(1^3)}\uparrow_{S_3}^{S_4}\right).$$

Give your answer in the form $V \cong \bigoplus_{\lambda \vdash 4} (\mathcal{S}^\lambda)^{\oplus m_\lambda}$, for certain non-negative multiplicities m_λ to be determined.

(c) Let $H \leqslant G$ be finite groups. Let χ be a character of G and ϕ a character of H. Show that

$$\chi \cdot (\phi\uparrow^G) = (\chi\downarrow_H \cdot \phi)\uparrow^G.$$

(d) Let $n \in \mathbb{N}$ with $n \geqslant 2$. You may assume that $\xi^{(n-1,1)} = \chi^{(n)} + \chi^{(n-1,1)}$.

(i) Let $\lambda \vdash n$. Using (c) or otherwise, show that

$$\langle \chi^\lambda \cdot \chi^\lambda, \chi^{(n-1,1)} \rangle = |\lambda^-| - 1.$$

[Hint: recall that $\xi^\lambda = \mathbb{1}\uparrow_{S_\lambda}^{S_n}$, where S_λ is a Young subgroup of type λ.]

(ii) Now suppose that n is prime and let $\alpha, \beta \vdash n$. Show that if $\chi^\alpha \cdot \chi^\beta$ is irreducible, then either α or β belongs to $\{(n), (1^n)\}$.

3

(a) Let $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ be a partition.

(i) Suppose $(i,j) \in \mathcal{Y}(\lambda)$. Show that

$$\{1,2,\ldots,h_{i,j}(\lambda)\} = \{h_{i,y}(\lambda) \mid j \leqslant y \leqslant \lambda_i\} \sqcup \{h_{i,j}(\lambda) - h_{x,j}(\lambda) \mid i < x \leqslant \lambda'_j\}.$$

(ii) Let $\mathbf{X} = \{h_1, \ldots, h_m\}$ be a β-set for λ. For $i \in \{1,2,\ldots,l(\lambda)\}$ and $h \in \mathbb{N}$, show that

$$h \in \mathcal{H}_i(\lambda) \iff h_i - h \geqslant 0 \text{ and } h_i - h \notin \mathbf{X},$$

where $\mathcal{H}_i(\lambda) = \{h_{i,j}(\lambda) \mid 1 \leqslant j \leqslant \lambda_i\}$.

[You may use without proof that $\mathcal{H}_i(\lambda) = \{1,2,\ldots,h_i\} \backslash \{h_i - h_j \mid i < j \leqslant m\}$.]

Deduce that if λ has a hook of length ef for some natural numbers e and f, then λ has a hook of length e.

(b) Let λ be any partition. Show that the number of odd hook lengths of λ minus the number of even hook lengths of λ is a triangular number, i.e. equal to $\binom{m}{2}$ for some $m \in \mathbb{N}$. What is m in terms of λ?

[Hint: first consider when λ is a 2-core partition.]

(c) (i) Compute the 4-quotient $Q_4(\lambda)$ of $\lambda = (3,1)$.

(ii) Calculate the 2-quotient tower $T^Q(\lambda)$ of $\lambda = (3,1)$.

(iii) Let e, k and n be natural numbers. Let $\lambda \vdash n$. Show that the sequence $Q_{e^k}(\lambda)$ is a permutation of $T^Q(\lambda)_k$, where $T^Q(\lambda)$ is the e-quotient tower of λ.

[Hint: induct on k.]

4 Fix a prime number p.

(a) Let n be a non-negative integer. Suppose the p-adic expansion of n is $n = \sum_{r=0}^{\infty} \alpha_r p^r$. That is, the digits α_r belong to $\{0, 1, \ldots, p-1\}$ for all $r \in \mathbb{N}_0$. Let $\lambda \vdash n$. Prove that

$$\mathbf{v}_p\big(\chi^\lambda(1)\big) = \frac{\sum_{r=0}^{\infty} |T^C(\lambda)_r| - \sum_{r=0}^{\infty} \alpha_r}{p-1},$$

where $T^C(\lambda)$ denotes the p-core tower of λ.

[You may use earlier results from the course without proof, provided they are stated clearly.]

(b) Define the function $d : \mathbb{N}_0 \to \mathbb{N}_0$ by setting $d(n)$ to be the sum of the digits in the p-adic expansion of n. In particular, for n as in part (a), this means $d(n) = \sum_{r=0}^{\infty} \alpha_r$.

Show that $d(x+y) \leqslant d(x) + d(y)$ for all non-negative integers x and y.

(c) Let λ be any partition. Prove that

$$\mathbf{v}_p\big(\chi^\lambda(1)\big) \geqslant \mathbf{v}_p\big(\chi^{C_p(\lambda)}(1)\big).$$

[Hint: recall that $T^C(\lambda)_r$ is the concatenation of $T^C(\lambda^{(j)})_{r-1}$ over $j \in \{0, 1, \ldots, p-1\}$, for each $r \geqslant 1$.]

END OF PAPER