UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 21 June, 2021 12:00 pm to 3:00 pm

## PAPER 159

## ALGEBRAIC SURFACES

***Before you begin please read these instructions carefully***

*Candidates have THREE HOURS to complete the written examination.*

*Attempt **ALL** questions.*
*There are **THREE** questions in total.*
*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** All surfaces appearing in this question will be smooth and projective.

What is the *Picard group* of a surface and what is its group operation? Define the intersection pairing between elements of the Picard group. Describe the Picard group and its intersection pairing on the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.

Let $\pi : X \to \mathbb{P}^2$ be a birational morphism and assume $\pi^{-1}$ is not defined everywhere. Prove that the Picard group of $X$ contains an element whose self-intersection is zero.

Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^n$ be a morphism with the property that there exists a line $\ell \subset \mathbb{P}^2$ such that $\varphi(\ell)$ is a point. Prove that $\varphi$ is constant.

Let $X$ be a surface and $C \subset X$ be a smooth curve with positive self-intersection. Construct a new surface $X'$ and a morphism $\pi : X' \to X$ such that $X'$ contains a curve $C'$ with negative self-intersection, and $\pi$ restricts to an isomorphism from $C'$ to $C$.

**2** All surfaces appearing in this question will be smooth, connected, and projective.

What is a *minimal surface*? Prove that an abelian surface is minimal. Prove that there exist infinitely many non-isomorphic minimal rational surfaces.

What is a $K3$ *surface*? Let $C$ be a smooth curve of genus $g$ on a $K3$ surface $X$. Calculate the self-intersection of $C$.

What is an *elliptically fibered surface?* Show that an elliptically fibered surface can have Kodaira dimension equal to $-\infty$. Construct an elliptically fibered $K3$ surface.

What is a *surface of general type*? Construct a general type surface that admits a finite degree 2 morphism to $\mathbb{P}^2$.

**3** All surfaces appearing in this question will be smooth and projective.

What is the *irregularity* of a surface? What is the *geometric genus* of a surface?

Let $X' \to X$ be a blowup of a surface at a point. Prove that the geometric genus of $X'$ coincides with that of $X$.

Let $X$ be a product of two curves of positive genus. Prove that $X$ is not isomorphic to a hypersurface in projective space.

What is the *Albanese variety*? Let $X$ be a surface and let $X \to \mathrm{Alb}(X)$ be the morphism to its Albanese variety. Assume that the image of $X$ is a smooth curve of genus $g$. Calculate the irregularity of $X$.

## END OF PAPER