UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 23 June, 2021 12:00 pm to 3:00 pm

PAPER 158

INFINITE GAMES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Infinite Games

In this question, work in ZF without the Axiom of Choice. Let M, X, and Y be arbitrary sets.

(i) Let $A \subseteq M^\omega$. Define what it means that A is *quasidetermined*.

(ii) Define the axiom $\mathsf{AC}_X(Y)$.

(iii) Write Φ_M for the statement "every quasidetermined set $A \subseteq M^\omega$ is determined". Specify sets X and Y such that $\mathsf{AC}_X(Y)$ is equivalent to Φ_M.

[You do not need to prove your claim.]

(iv) Let $A \subseteq \mathbb{R} \times \mathbb{R}$ and $\mathrm{p}A := \{y \in \mathbb{R}\,;\, \exists x \in \mathbb{R}((x, y) \in A)\}$ be its *projection*. A function $f : \mathrm{p}A \to \mathbb{R}$ is called a *uniformisation of* A if for all $x \in \mathrm{p}A$, we have that $(f(x), x) \in A$. Show that $\mathsf{AD}_{\mathbb{R}}$ implies that every set $A \subseteq \mathbb{R} \times \mathbb{R}$ has a uniformisation.

2 Infinite Games

In this question, work in ZFC with the Axiom of Choice. Let $A \subseteq \omega^\omega$ and let X and Y be arbitrary sets. Let κ be any infinite cardinal and κ^+ be its cardinal successor.

Recall that a set $A \subseteq \omega^\omega$ is called X-*Suslin* if there is a tree $T \subseteq (X \times \omega)^{<\omega}$ such that for all $x \in \omega^\omega$, we have that $x \in A$ if and only if there is a $y \in X^\omega$ such that $(y, x) \in [T]$.

(i) Suppose that there is an injection $f : X \to Y$. Show that A is X-Suslin, then A is Y-Suslin.

(ii) Prove that every set $A \subseteq \omega^\omega$ is $2^{\aleph_0}$-Suslin.

(iii) Show that every κ^+-Suslin set is a union of κ^+ many κ-Suslin sets.

(iv) We write Ψ_κ for the statement "there is a set that is not κ-Suslin". Prove both of the following implications:

$$\Psi_{\aleph_2} \Longrightarrow 2^{\aleph_0} > \aleph_2 \Longrightarrow \Psi_{\aleph_1}.$$

[You may use the fact that all analytic sets have the perfect set property.]

3 Infinite Games

In this question, work under the assumption that ZFC is consistent. You may use the fact that under this assumption, the theory ZFC + CH is also consistent. In this question, an *explanation* of a statement consists of a list of results stated in the lectures, stated clearly and correctly, and a brief argument that these results imply the statement. You do not need to give a proof of any of the results in your list.

(i) Explain the following statement:

> *If M is a projectively well-ordered inner model and all projective sets are determined, then $\aleph_1^M$ is a countable ordinal.*

Give precise definitions of the concepts "projectively well-ordered" and "$\aleph_1^M$" occurring in the statement.

(ii) Explain why it is not possible that ZFC proves the statement "All $\aleph_1$-Suslin sets are determined."

4 Infinite Games

In this question, work in ZF + AD without the Axiom of Choice.

(i) Consider the following three games played as follows

$$\begin{array}{l|cccccccccl} \text{Player I} & x_0 & & x_1 & & x_2 & & x_3 & & \cdots & \rightsquigarrow x \in \omega^\omega \\ \text{Player II} & & y_0 & & y_1 & & y_2 & & y_3 & \cdots & \rightsquigarrow y \in \omega^\omega \end{array}$$

and determine which of the two players has a winning strategy. Justify your claim.

[You may use theorems proved in the lectures without proof, provided that you state them clearly and correctly.]

(a) If $y \notin \mathrm{WF}$, player I wins; if $y \in \mathrm{WF}$, but $x \notin \mathrm{WF}$, player II wins; if $x, y \in \mathrm{WF}$, then player II wins if $\|x\| < \|y\|$.

(b) If we have $y_n \leqslant x_n$ for all n, then player I wins; otherwise, player II wins if $y \notin \mathrm{WF}$.

(c) If $x \notin \mathrm{WF}$, then player I wins if and only if $x = y$; if $x \in \mathrm{WF}$, then player II wins if and only if $y \in \mathrm{WF}$ and $\|x\| < \|y\|$.

(ii) Assume that there is a surjection $\pi : \omega^\omega \to \aleph_2$ and a sequence $\{g_\xi \,;\, \xi < \omega_2\}$ such that each g_ξ is a surjection from ω^ω onto $\wp(\xi)$. Show that there is a surjection from $\omega^\omega \to \wp(\aleph_2)$.

[Hint. Use Friedman-Moschovakis games.]

END OF PAPER