UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2021 12:00 pm to 2:00 pm

PAPER 157

COMPLEX DYNAMICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In this question, $f(z) \in \mathbb{C}(z)$ is a rational map of degree at least 2.

(a) Define what is meant by a *normal family* of holomorphic functions, and state Montel's theorem.

(b) Suppose that E is a closed subset of the Riemann sphere satisfying $f^{-1}(E) = E$. Prove that either E contains at most two elements and E is contained in the Fatou set, or $J(f) \subset E$.

(c) Prove that the Julia set of f contains no isolated points.

2

(a) Prove that an attracting fixed point of a rational map attracts a critical point.

(b) State the classification theorem of fixed Fatou components for a rational map of degree $d \geqslant 2$, and describe how each type of component interacts with the set of forward orbits of critical points of the map [without proofs].

(c) Prove that for each $d \geqslant 2$, there exists a degree d rational map which has Julia set equal to the Riemann sphere $\hat{\mathbb{C}}$.

(d) Prove that for each $d \geqslant 2$, there exists a degree d rational map with exactly one Fatou component.

3

(a) Define the Böttcher coordinate ϕ at infinity for a polynomial map of degree $d \geqslant 2$, and the dynamically natural maximal domain to which its modulus $|\phi|$ extends. Define the Green's function of a polynomial, and explain how it relates to the Böttcher map.

(b) Prove that if a polynomial of degree at least 2 has connected Julia set, then all finite critical points are contained in the filled Julia set.

(c) Describe briefly and without proofs how the Böttcher coordinate at infinity is used to prove that the Mandelbrot set is connected.

4

(a) Define the *residue index* of a holomorphic map f at a fixed point z_0 of f. Suppose that $f(z)$ is a quadratic rational map with distinct fixed points z_0, z_1, and z_2, with multipliers λ_0, λ_1, and λ_2 respectively. Show that $\lambda_i \lambda_j \neq 1$ whenever $i \neq j$.

(b) Let $g(z)$ be a rational map of degree at least 2, with a parabolic fixed point at $z = 0$ of multiplier 1. Let A be the parabolic basin of some attracting vector for the parabolic fixed point 0. Prove that the boundary ∂A is contained in the Julia set $J(g)$.

(c) Given $c \in \mathbb{C}$, write $f_c(z) = z^2 + c$. Prove that

$$\{n \in \mathbb{N} : \exists c \in \mathbb{C} \text{ such that } f_c \text{ has a parabolic cycle of period } n\}$$

is an infinite set. [Hint: it may simplify your argument to prove that this set contains all of the prime numbers.]

END OF PAPER