UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 24 June, 2021 12:00 pm to 2:00 pm

PAPER 156

MAPPING CLASS GROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let S be a closed, orientable surface.

(a) State without proof a condition on collections of simple closed curves

$$\{\alpha_1, \ldots, \alpha_n\}, \{\beta_1, \ldots, \beta_n\}$$

that guarantees that, if α_i is homotopic to β_i for each i, then there is an ambient isotopy of S taking α_i to β_i for all i.

(b) Define the *structure graph* of a collection of simple closed curves $\{\alpha_i\}$ on S. State and prove the Alexander method, using your condition from part (a).

(c) Let T^2 be the torus. By considering a pair of essential simple closed curves on T^2, and using the Alexander method, show that the centre of $\mathrm{Mod}(T)$ has order two. [*State carefully any results about Dehn twists that you use.*]

2 Let S be a connected, orientable surface of finite type and let G be its fundamental group.

(a) Explain briefly why there is a natural homomorphism

$$p : \mathrm{Mod}(S) \to \mathrm{Out}(G)\,,$$

where $\mathrm{Out}(G)$ denotes the outer automorphism group of G.

(b) The *commutator subgroup* $[G, G]$ of a group G is defined to be the intersection of the kernels of the set of homomorphisms from G to abelian groups. The *abelianisation* of G is the abelian group

$$G_{\mathrm{ab}} := G/[G, G]\,.$$

Prove that the map $q : \mathrm{Out}(G) \to \mathrm{Aut}(G_{\mathrm{ab}})$ defined by

$$q([A])(g[G, G]) = A(g)[G, G]$$

is a well-defined group homomorphism.

(c) When S is the punctured torus T^2_*, recall that the fundamental group $G = \pi_1(T^2_*)$ is the free group on two generators $\langle \alpha, \beta \rangle$, and its abelianisation G_{ab} is isomorphic to $\mathbb{Z}^2$. By considering the maps p and q, show that $\mathrm{Out}(G)$ is infinite.

[*You may use without proof that the matrices*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \; \textit{and} \; \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

generate $SL_2(\mathbb{Z})$.]

(d) Give an example where the image $p(\mathrm{Mod}(S))$ is a subgroup of infinite index in $\mathrm{Out}(\pi_1(S))$.

3 (a) Let S be a connected, oriented, hyperbolic surface of finite type. Define the *pure* mapping class group of S. Let S_* denote S with an additional puncture. Describe the *point pushing map* in terms of Dehn twists and state the Birman exact sequence carefully.

(b) For $S = S_{0,3,0}$, the 3-punctured sphere, prove that $\mathrm{PMod}(S)$ is trivial. [If you use a result classifying simple proper arcs, you should prove it.]

(c) Let $\Sigma = S_{0,5,0}$, the 5-punctured sphere. Prove that $\mathrm{PMod}(\Sigma)$ has a free normal subgroup K and a free subgroup F such that $F \cap K$ is trivial and, furthermore, $\mathrm{PMod}(\Sigma) = FK$. (That is, every element of $\mathrm{PMod}(\Sigma)$ is of the form fk for $f \in F$ and $k \in K$.) What is the minimal size of a generating set for F? What about K?

[*Hint: You may use the following fact without proof. If F is a free group with minimal generating set of cardinality $r < \infty$, and G is any group with a generating set of cardinality r, then any surjective homomorphism $G \to F$ is an isomorphism.*]

4 Let S be the closed, orientable surface of genus 2.

(a) Define the complex of curves $C(S)$, and prove that it is connected.

(b) Consider a set of essential simple closed curves $\alpha_1, \ldots, \alpha_n$ in minimal position. What does it mean for the set $\alpha_1, \ldots, \alpha_n$ to *fill* S?

(c) The *distance* between two vertices in $C(S)$ is the smallest number of edges in a path in the 1-skeleton between them. Exhibit a pair of vertices of $C(S)$ at distance strictly greater than 2. [Any explicit curves that you use should be specified carefully.]

END OF PAPER