UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 10 June, 2021 12:00 pm to 3:00 pm

PAPER 155

METRIC EMBEDDINGS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the notions of *coarse embedding* and *uniform coarse embedding.*

Let H_n denote the Hamming cube of dimension n. Show that $\{H_n : n \in \mathbb{N}\}$ uniformly coarsely embeds into L_1. Does $\{H_n : n \in \mathbb{N}\}$ uniformly coarsely embed into L_2? Justify your answer.

Define the *expanding constant* of a (finite) graph. What is a *family of expanders*?

Let $G = (V, E)$ be a graph on n vertices with expanding constant $h = h(G) > 0$. Show that for a function $f : V \to \mathbb{R}$ with median M, the following inequality holds:

$$\sum_{x,y \in V} a_{x,y}|f(x) - f(y)| \geqslant 2h \sum_{x \in V} |f(x) - M|$$

where $A = (a_{x,y})_{x,y \in V}$ is the adjacency matrix of G. Deduce that

$$\sum_{x,y \in V} a_{x,y}|f(x) - f(y)| \geqslant \frac{h}{n} \sum_{x,y \in V} |f(x) - f(y)|$$

and hence obtain a Poincaré inequality for L_1-valued functions on V. Deduce the following lower bound of the L_1-distortion of G if G is d-regular for some $d \geqslant 3$:

$$c_1(G) \geqslant \frac{h}{2d \log d} \log\left(\frac{n}{2} - 1\right)$$

Prove that a family of expanders does not uniformly coarsely embed into L_1. Does the same hold if we replace L_1 with L_2? Justify your answer.

[*Throughout this question you may use lower bounds on distortion in terms of Poincaré ratios.*]

2

Let X and Y be Banach spaces. What does it mean to say that X is *finitely representable in* Y? What does it mean to say that X is *superreflexive*?

Prove that a Banach space X is reflexive if and only if for all $\theta > 0$, for all sequences (x_i) in the closed unit ball B_X, there exist $n \in \mathbb{N}$, $y \in \operatorname{conv}\{x_i : 1 \leqslant i \leqslant n\}$ and $z \in \operatorname{conv}\{x_i : i > n\}$ such that $\|y - z\| < \theta$. [You may use the Principle of Local Reflexivity without proof.]

Prove that a Banach space X is superreflexive if and only if for all $\theta > 0$, there exists $N \in \mathbb{N}$ such that every sequence $(x_i)_{i=1}^N$ in B_X has the following property: there exist $n \in \mathbb{N}$ with $1 \leqslant n < N$, $y \in \operatorname{conv}\{x_i : 1 \leqslant i \leqslant n\}$ and $z \in \operatorname{conv}\{x_i : n+1 \leqslant i \leqslant N\}$ such that $\|y - z\| < \theta$. [You may assume without proof that any ultrapower of a Banach space Z is finitely representable in Z.]

State a purely metric condition on a Banach space that is equivalent to superreflexivity. Show that the condition is sufficient. [You may assume any other characterization of superreflexivity from the course as well as results about embeddings of diamond graphs into $\mathbb{R}^n$ with standard norms.]

3

State Bourgain's Embedding Theorem.

Prove that there is a constant $C > 0$ such that for all $q, n \in \mathbb{N}$, $q \geqslant 2$, every n-point metric space embeds into ℓ_∞^k with distortion at most $\alpha = 2q - 1$ for some $k \leqslant Cqn^{1/q} \log n$. Deduce that $c_2(n) = O(\log^2 n)$.

Show that there is a constant $c > 0$ such that if G is a d-regular graph on n vertices with expanding constant h that embeds into ℓ_∞^k with distortion at most α, then $k \geqslant n^{c/\alpha}$, where $d \geqslant 3$ and c depends only on d and h. [You may assume that for such a graph G, the L_p-distortion $c_p(G) \geqslant C\frac{1}{p} \log n$ for all $1 < p < \infty$, where C is a constant depending only on d and h.]

4

State the Johnson–Lindenstrauss lemma.

Let E, X be normed spaces with $n = \dim E < \infty$ and let $\delta \in (0, 1/3)$. Show that the unit sphere S_E of E has a δ-net $S \subset S_E$ of size $|S| \leqslant \left(\frac{3}{\delta}\right)^n$. Show that if $T\colon E \to X$ is a linear map such that $1 - \delta \leqslant \|Tx\| \leqslant 1 + \delta$ for all $x \in S$, then T is injective with $\|T\|\|T^{-1}\| \leqslant \frac{1+\delta}{1-3\delta}$.

Let Y be a random variable with the standard normal distribution. (Thus, $Y \sim N(0, 1)$ has pdf $\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$.) It is known that for some constant $C > 0$ the following inequalities hold:

$$\mathbb{E} e^{u(|Y|-\beta)} \leqslant e^{Cu^2} \qquad \text{and} \qquad \mathbb{E} e^{-u(|Y|-\beta)} \leqslant e^{Cu^2}$$

for all $u \geqslant 0$ where $\beta = \sqrt{\frac{2}{\pi}}$. Prove the first of these inequalities. [Hint: Note that $\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2}\, dy = \frac{1}{2}$ and $1 + x \leqslant e^x$ for all $x \in \mathbb{R}$.]

Let $n, k \in \mathbb{N}$ and $\delta \in (0, 1)$. Let $T\colon \ell_2^n \to \ell_1^k$ be the random linear map given by

$$\big(Tx\big)_i = \frac{1}{\beta k} \sum_{j=1}^{n} Z_{i,j} x_j \qquad \text{for } x = (x_j)_{j=1}^n \in \ell_2^n \text{ and } 1 \leqslant i \leqslant k$$

where the $Z_{i,j}$, $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$, are independent standard normal distributions. Show that for every $x \in \mathbb{R}^n$ and $\delta \in (0, 1)$ we have

$$\mathbb{P}\Big[(1-\delta)\|x\|_2 \leqslant \|Tx\|_1 \leqslant (1+\delta)\|x\|_2\Big] \geqslant 1 - 2e^{-c\delta^2 k}$$

where $c > 0$ is an absolute constant independent of k, n, δ.

Show that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that whenever $k \geqslant C_\varepsilon n$, there is a linear embedding $T\colon \ell_2^n \to \ell_1^k$ of distortion $\|T\|\|T^{-1}\| < 1 + \varepsilon$.

Show that there is a constant $C > 0$ such that every n-point metric space embeds into ℓ_1^k with distortion at most $C \log n$, where $k \leqslant C \log n$.

END OF PAPER