UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2021 12:00 pm to 3:00 pm

PAPER 154

INTRODUCTION TO NONLINEAR ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **THREE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The KdV equation

We consider the KdV equation for $p = 2, 3, 4, 5$:

$$(KdV)\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^p) = 0, \\ u_{|t=0} = u_0(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \quad u(t,x) \in \mathbb{R}. \end{cases}$$

You are given that (KdV) is locally well posed in $H^1(\mathbb{R})$ with the blow up criterion

$$T < +\infty \text{ iff } \lim_{t \to T} \|u(t)\|_{H^1} = +\infty.$$

In the sequel, you do not need to give justifications for dropping boundary terms when integrating by parts.

1. Show that the L^2 norm is conserved by the flow.
2. Show that the energy
$$E(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1} dx$$
is conserved by the flow.
3. Show that all H^1 solutions are global for $p < 5$.
4. Let Q be the ground state solution to $Q'' - Q + Q^p = 0$. Given $c > 0$, compute Q_c in terms of Q and c such that $u(t,x) = Q_c(x - ct)$ solves (KdV).
5. Discuss the orbital stability of Q_c in the energy space $H^1(\mathbb{R})$ for $p < 5$.

2 The Hartree equation

Let $N = 3, 4$. We consider the gravitational Hartree equation

$$(Hartree)\begin{cases} i\partial_t u + \Delta u - \phi u = 0, \\ \phi(t,x) = -\frac{1}{C_N|x|^{N-2}} \star |u(t,x)|^2, \\ u_{|t=0} = u_0(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \quad u(t,x) \in \mathbb{C}, \end{cases} \tag{1}$$

where C_N is the area of the unit sphere of $\mathbb{R}^N$. We recall that ϕ solves the Laplace equation

$$\Delta\phi = |u|^2.$$

You are given that (Hartree) is locally well posed in $H^1(\mathbb{R}^N)$ with blow up critetrion

$$T < +\infty \quad \text{iff} \quad \lim_{t\to T} \|u(t)\|_{H^1} = +\infty.$$

In the sequel, you do not need to give justifications for dropping boundary terms when integrating by parts.

1. Show that $\int_{\mathbb{R}^N} |u(t,x)|^2 dx$ is conserved.
2. Show that the energy $E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx - \frac{1}{4}\int_{\mathbb{R}^N} |\nabla \phi(t,x)|^2 dx$ is conserved.
3. Show that all solutions are global in dimension $N = 3$.
4. Let $N = 4$. Show that

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 dx = 16E(u).$$

Are all solutions global in time?

3 Critical Non Linear Schrödinger equation

We work in dimension $d = 2$.

1. Let $M > 0$ and $\eta > 0$. We let $\mathcal{A}(M) = \{u \in H^1(\mathbb{R}^2), \ \ \int_{\mathbb{R}^2} |u|^4 = M\}$. Show that the minimization problem
$$I(M) = \inf_{u \in \mathcal{A}(M)} \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 + \int_{\mathbb{R}^2} |u|^2 + \frac{\eta}{4} \int_{\mathbb{R}^2} |x|^2 |u|^2 \right\}$$
is attained.

2. Write down (without justification) the Euler-Lagrange equation satisfied by a positive minimizer and prove that there exists a non trivial solution $P_\eta \in H^1(\mathbb{R}^2)$ to
$$\Delta P_\eta - P_\eta - \frac{\eta}{4}|x|^2 P_\eta + P_\eta^3 = 0. \tag{1}$$

3. Solve the dynamical system
$$\left| \begin{array}{l} \frac{ds}{dt} = \frac{1}{\lambda_\eta^2}, \quad \frac{db_\eta}{ds} + b_\eta^2 = -\eta, \quad b_\eta = -\frac{1}{\lambda_\eta}\frac{d\lambda_\eta}{ds} \\ \lambda_\eta(t=-1) = \sqrt{1+\eta}, \quad b_\eta(t=-1) = 1. \end{array} \right.$$
(Hint: compute $\frac{d}{ds}\left(\frac{\sqrt{b_\eta^2+\eta}}{\lambda_\eta}\right)$).

4. Let $y = \frac{x}{\lambda_\eta(t)}$. Compute $\gamma_\eta(t)$ such that
$$u_\eta(t,x) = \frac{1}{\lambda_\eta(t)}\left(P_\eta(y) e^{-i\frac{b_\eta(t)|y|^2}{4}}\right) e^{i\gamma_\eta(t)}$$
solves the L^2 critical focusing (NLS) equation.

5. Let $S(t)$ be the free Schrödinger semi group. Show that the sequence of functions
$$t \mapsto \int_{-1}^{t} S(-s)\left(u_\eta(s,y)|u_\eta(s,y)|^2\right) ds$$
has a strong L^2 limit as $t \to +\infty$.
(Hint: use Strichartz and the bound $\|u_\eta\|_{L^4_{[-1,+\infty)}L^4_x} < +\infty$ to show that it is Cauchy in L^2).
Conclude that there exists $u_\eta^\infty \in L^2$ such that
$$u_\eta - S(t)u_\eta^\infty \to 0 \ \text{ in } \ L^2 \text{ as } \ t \to +\infty.$$

END OF PAPER