UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 15 June, 2021 12:00 pm to 3:00 pm

PAPER 151

PROFINITE GROUPS AND GROUP COHOMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) (i) State an explicit description of the inverse limit of an inverse system of sets.

(ii) Show that an inverse limit of an inverse system of non-empty finite sets is non-empty.

(b) Let $\{G_j\}_{j\in J}$ be an inverse system of finite groups indexed over an inverse system J. Let $G = \varprojlim G_j$ and let $p_j \colon G \to G_j$ be the projection map for $j \in J$.

(i) Show that G, with its standard topology, is a topological group.

(ii) Deduce that, for a fixed element $h \in G$, the function

$$c_h \colon G \to G, \quad g \mapsto ghg^{-1}$$

is continuous.

(iii) Show that the conjugacy class

$$\mathrm{CCl}(h, G) = \{ghg^{-1} \mid g \in G\}$$

of an element $h \in G$ is a closed set.

(iv) Let $g, h \in G$. Show that g and h are conjugate in G if and only if $p_j(g)$ and $p_j(h)$ are conjugate in G_j for all $j \in J$.

(c) Let Γ be an abstract group. We say that Γ is *conjugacy separable* if for every $g, h \in \Gamma$ such that g is not conjugate to h in Γ there exists a homomorphism $f \colon \Gamma \to Q$ from Γ to a finite group Q such that $f(g)$ is not conjugate to $f(h)$ in Q.

Assume that Γ is residually finite, and identify Γ with its image under the canonical inclusion $\Gamma \hookrightarrow \widehat{\Gamma}$.

(i) Let $\gamma \in \Gamma$. Show that $\mathrm{CCl}(\gamma, \widehat{\Gamma})$ is the closure of $\mathrm{CCl}(\gamma, \Gamma)$ in $\widehat{\Gamma}$.

(ii) Show that Γ is conjugacy separable if and only if

$$\mathrm{CCl}(\gamma, \widehat{\Gamma}) \cap \Gamma = \mathrm{CCl}(\gamma, \Gamma)$$

for every $\gamma \in \Gamma$.

2 (a) Let $\{G_j\}_{j\in J}$ be an inverse system of finite groups. Let $G = \varprojlim G_j$ and let $p_j \colon G \to G_j$ be the projection maps. We say that $S \subseteq G$ is a *topological generating set* for G if the subgroup of G generated by S is dense in G.

(i) State a criterion for S to be a topological generating set of G, in terms of the inverse system $\{G_j\}$.

(ii) Let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let $\alpha \in \mathbb{Z}_p$. Show that $\{\alpha\}$ is a topological generating set for the (additive) group $\mathbb{Z}_p$ if and only if α is not mapped to 0 under the projection map $\mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$.

(iii) Show that $\{\alpha\}$ is a topological generating set for $\mathbb{Z}_p$ if and only if there exists $\beta \in \mathbb{Z}_p$ such that $\alpha\beta = 1$ (where the multiplication is the ring multiplication of $\mathbb{Z}_p$).

(b) (i) Let A be a finitely generated abelian group. Show that $\widehat{A} \cong \widehat{\mathbb{Z}}$ if and only if $A \cong \mathbb{Z}$.

(ii) Let Γ be a finitely generated abstract group. Show that $\widehat{\Gamma}$ is not isomorphic to $\mathbb{Z}_p$ for any prime p.

(c) Let π be a set of prime numbers. Define a subgroup of $\mathbb{Q}$ by

$$A_\pi = \left\{ \frac{m}{n} \text{ such that } n = p_1^{e_1} \cdots p_k^{e_k} \text{ for } p_1, \ldots, p_k \in \pi \right\}.$$

(i) Let p be a prime number and let $\neg p$ denote the set of all primes not equal to p. Define an injective homomorphism $A_{\neg p} \hookrightarrow \mathbb{Z}_p$.

[You need not give a detailed justification that your function is a homomorphism, but should give a reason that it is injective.]

(ii) Deduce that A_π is residually finite unless π is equal to the set of all primes.

(iii) Let $q \neq p$ be a prime. Show that the only homomorphism from $A_{\neg p}$ to $\mathbb{Z}/q\mathbb{Z}$ is the trivial homomorphism.

(iv) Deduce that a finite group is a finite quotient of $A_{\neg p}$ if and only if it is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ for some n.

3 (a) Let $G = \varprojlim_{j \in J} G_j$ be a profinite group, where $\{G_j\}_{j \in J}$ is an inverse system of finite groups.

(i) Suppose $\mathcal{U}$ is a neighbourhood base of $1 \in G$, all of whose elements are normal subgroups of G. Show that $G \cong \varprojlim_{U \in \mathcal{U}} G/U$.

[Standard properties of inverse limits may be freely used.]

(ii) Suppose that G is topologically finitely generated. Show that G has only finitely many open subgroups of index n, for $n \in \mathbb{N}$.

(iii) Let G_n be the intersection of all open subgroups of G of index at most n. Show that if G is topologically finitely generated then $G \cong \varprojlim G/G_n$.

(b) Let G be a topologically finitely generated profinite group.

(i) Define the *Hopf property* for topological groups.

(ii) Show that G has the Hopf property.

(c) Let $m \in \mathbb{N}$.

(i) Let H be a finite group. Show that the function $f_m \colon H \to H$ defined by $f_m(x) = x^m$ is bijective if and only if m is coprime to $|H|$.

(ii) Let $G = \varprojlim_{j \in J} G_j$ be a profinite group, where $\{G_j\}_{j \in J}$ is an inverse system of finite groups. Let $p_j \in G \to G_j$ be the projection maps. Let $g \in G$. Show that there exists $x \in G$ such that $x^m = g$ if and only if for all j, there exists $x \in G_j$ such that $p_j(g) = x^m$.

[Standard properties of inverse limits may be freely used.]

(iii) Deduce that if m is coprime to $|G_j|$ for all j then the continuous function $f_m \colon G \to G$, $x \mapsto x^m$ is bijective.

4 (a) Let G be a finite group and let M be a G-module.

(i) Define the *cochain groups* $C^n(G, M)$ for $n \geqslant 0$ and the differentials $d^n \colon C^{n-1}(G, M) \to C^n(G, M)$ for $n \geqslant 1$.

(ii) What is meant by a *crossed homomorphism* $\phi \colon G \to M$? State and prove a characterisation of the first cohomology group $H^1(G, M)$ in terms of crossed homomorphisms.

(iii) Let $\phi \in Z^2(G, M) = \ker d^3$. Define

$$\psi(g) = \sum_{\gamma \in G} \gamma^{-1} \phi(\gamma, g).$$

Show that $d^2\psi = |G| \cdot \phi$.

(iv) Deduce that $H^2(G, \mathbb{Q}) = 0$ (where $\mathbb{Q}$ has trivial G-action).

(v) Find $H^1(G, \mathbb{Q})$ and $H^0(G, \mathbb{Q})$.

(b) Let G be a finite group.

(i) Consider a short exact sequence of G-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

What is the relationship between the families of cohomology groups $H^\bullet(G, M_1)$, $H^\bullet(G, M_2)$ and $H^\bullet(G, M_3)$?

[You need not give definitions of any homomorphisms involved.]

(ii) Show that $H^2(G, \mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z})$.

(c) Let $D_n = \langle a, b \mid a^2 = 1, b^n = 1, ab = b^{-1}a \rangle$ denote the dihedral group of order $2n$.

(i) Compute $H^1(D_n, \mathbb{Z})$ and $H^2(D_n, \mathbb{Z})$.

(ii) Let M denote the D_n-module whose underlying group is $\mathbb{Z}$, and with D_n-action defined by $a \cdot n = -n, b \cdot n = n$. Define a map $\phi \colon D_n \to M$ by

$$\phi(a^k b^l) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \end{cases}$$

Using the function ϕ, show that $H^1(D_n, M) \neq 0$.

[You may use the fact that the elements $a^k b^l$ for $k = 0, 1$ and $0 \leqslant l \leqslant n-1$ comprise all elements of D_n.]

5 (a) Let $f: G \to H$ be a group homomorphism and let M be an H-module. Give M the structure of a G-module by setting $g \cdot m = f(g) \cdot m$ for $g \in G$ and $m \in M$.

(i) Given an extension

$$1 \to M \to E \xrightarrow{\pi} H \to 1$$

define a 2-cochain ϕ representing the extension E and prove that ϕ is a cocycle.

(ii) Prove that the group

$$E' = \{(e, g) \in E \times G \mid \pi(e) = f(g)\}$$

is an extension of G by M representing the cohomology class $f^*([\phi])$, where $[\phi] \in H^2(G, M)$ denotes the cohomology class represented by a cocycle ϕ.

(iii) Show that if f is an injection then the natural map $E' \to E$ is injective.

(b) Let Γ be an abstract group, let $\iota: \Gamma \to \widehat{\Gamma}$ be the canonical map and let G be a profinite group. Show that a homomorphism $f: \Gamma \to G$ extends uniquely to a continuous homomorphism $\hat{f}: \widehat{\Gamma} \to G$ such that $\hat{f}\iota = f$.

(c) Let Γ be a residually finite abstract group and let $\iota: \Gamma \to \widehat{\Gamma}$ be the canonical inclusion. Let M be a finite $\widehat{\Gamma}$-module. Consider the natural map $\iota^*: H^2(\widehat{\Gamma}, M) \to H^2(\Gamma, M)$.

(i) Let $\zeta \in \operatorname{im} \iota^*$ and let E'_ζ denote the extension of Γ by M corresponding to ζ. Show that E'_ζ is residually finite.

[You may assume that any extension of $\widehat{\Gamma}$ by M is a profinite group.]

(ii) Show that ι^* is always injective.

END OF PAPER