UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 25 June, 2021 12:00 pm to 2:00 pm

## PAPER 150

## ANALYTIC NUMBER THEORY

### Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

In this question you may use without proof the upper bounds $\psi(x) \ll x$ and $\pi(x) \ll x/\log x$.

(a) Show that there exists a constant $c$ such that

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c + O(1/\log x),$$

where the summation is restricted to prime numbers.

(b) Let $\omega(n)$ count the number of distinct prime divisors of $n$. By estimating the sum $\sum_{n \leqslant x}(\omega(n) - \log\log x)^2$, or otherwise, show that for all large $x$

$$\#\{n \leqslant x : |\omega(n) - \log\log n| > (\log\log n)^{3/4}\} = o(x).$$

**2** In this question you may assume any standard properties of the Gamma function $\Gamma(s)$ without proof, provided you state them clearly. We use $\sigma$ and $t$ to denote the real and imaginary parts respectively of the complex variable $s$.

(a) Define, with justification, the Riemann zeta function $\zeta(s)$ as a meromorphic function in the half-plane $\sigma > 0$.

(b) State the functional equation for $\zeta(s)$ and sketch a proof, using any method you prefer. You do not need to justify every technical detail, such as when interchanging a sum and an integral, but should focus on describing the main ideas in the proof.

(c) Show that, uniformly for $-2 \leqslant \sigma \leqslant 2$ and $t \geqslant 4$,

$$|\zeta(s)| \asymp t^{1/2-\sigma}|\zeta(1-s)|.$$

[You may use without proof Stirling's approximation that

$$\Gamma(s) = \sqrt{2\pi}s^{s-1/2}e^{-s}(1 + O(1/|s|))$$

uniformly for $-4 \leqslant \sigma \leqslant 4$ and $|t| \geqslant 4$.]

**3** Let $\sigma$ and $t$ denote the real and imaginary parts respectively of the complex variable $s$. In this question you may assume without proof that if $|t| \geqslant 7/8$ and $5/6 \leqslant \sigma \leqslant 2$ then

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\log(|t|+4))$$

where the sum is over all zeros $\rho$ of $\zeta(s)$ in the region $|\rho - (3/2 + it)| \leqslant 5/6$. You may assume without proof any standard properties of $\zeta(s)$ provided you state them clearly.

(a) Explain why $\zeta(s) \neq 0$ for $\sigma > 1$.

(b) Prove that there is a constant $c > 0$ such that

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma \geqslant 1 - \frac{c}{\log|t|} \quad \text{and} \quad |t| \geqslant 4.$$

[You may find the fact that $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geqslant 0$ useful.]

(c) Show that if $\sigma > 1 - c/2\log|t|$, where $c$ is the constant from part (b), and $|t| \geqslant 8$, then

$$\frac{\zeta'}{\zeta}(s) \ll \log|t|.$$

**4**

(a) Suppose $x$ is not an integer and $T \geqslant 1$. State an explicit formula with error term expressing $\psi(x)$ in terms of the zeros $\rho$ of $\zeta(s)$ with $\rho = \beta + i\gamma$ where $0 \leqslant \beta \leqslant 1$ and $|\gamma| \leqslant T$.

(b) Prove that the Riemann hypothesis is equivalent to the estimate

$$\psi(x) = x + O_\epsilon(x^{1/2+\epsilon})$$

for any $\epsilon > 0$.

[*You may use without proof Landau's lemma and any standard properties of* $\zeta(s)$ *or the distribution of its zeros provided you state them clearly.*]

## END OF PAPER