UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 15 June, 2021 12:00 pm to 2:00 pm

PAPER 146

SYMPLECTIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***TWO*** *questions.*
There are ***THREE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the symplectic blow up of a standard symplectic ball at a point.

Define a Lefschetz pencil on a compact orientable four-manifold X. Find a formula for the Euler characteristic of X in terms of the genus of a smooth fibre, the number of base points, and the number of critical points.

Equip $\mathbb{CP}^2$ with the standard symplectic structure. Suppose there is a Lefschetz pencil on $\mathbb{CP}^2$ whose smooth fibres are symplectic. What are all the possible values for the genus of such a smooth fibre? Justify your answer.

2

Define what is meant by a compatible almost complex structure on a symplectic manifold (M, ω_M), and show that the space of compatible almost complex structures on (M, ω_M) is non-empty. Suppose that (N, ω_N) is a compact symplectic manifold and that there exists a symplectic embedding $\iota : N \to M$. Show that there exists a compatible almost complex structure J on M such that $J(TN) = TN$.

Suppose X is a Kaehler manifold of real dimension four, and $C_1, C_2 \subset X$ are compact holomorphic submanifolds intersecting transversally in a single point. Show that there exists an embedded symplectic submanifold $\Sigma \subset X$ such that $[\Sigma] = [C_1] + [C_2] \in H_2(X, \mathbb{Z})$. [*You may assume there exist local holomorphic coordinates* (x, y) *near the intersection point such that* $C_1 = \{x = 0\}$ *and* $C_2 = \{y = 0\}$.]

Give examples of such X, C_1, C_2 where:

(a) there exists a Σ and a compatible almost complex structure J on X such that all of C_1, C_2 and Σ are almost complex;

(b) there cannot exist a Σ and a compatible almost complex structure J on X such that all of C_1, C_2 and Σ are almost complex.

3

Given a symplectic manifold (M, ω), define the Hamiltonian vector field associated to a smooth family of functions $\{f_t\}_{t\in[0,1]} \in C^\infty(M)$, and show that its flow, where defined, acts by symplectomorphisms. Prove that for any $v \in \mathbb{R}^2$, $r > 0$ and $\epsilon > 0$, there is a compactly supported symplectomorphism of $(\mathbb{R}^2, \omega_0)$ which takes $B(r)$ to $B(r) + v$ by translation, and is the identity outside $B(r + |v| + \epsilon)$.

Let (X^4, ω_X) be symplectic. Let Σ_g denote the smooth surface of genus g. Given Lagrangian embeddings $\sigma_1 : \Sigma_{g_1} \to X$ and $\sigma_2 : \Sigma_{g_2} \to X$ intersecting transversally at a single point, construct a Lagrangian embedding $\sigma : \Sigma_{g_1+g_2} \to X$.

For any $l \geqslant 1$, show that we can find a smooth map $\iota : T^2 \to \mathbb{R}^4$ such that $\iota^*\omega_0 = 0$ and $\iota(T^2)$ self-intersects cleanly in l copies of S^1. By using a careful perturbation of ι, or otherwise, show that the closed connected non-orientable surface of Euler characteristic $-4l$ admits a Lagrangian embedding into $(\mathbb{R}^4, \omega_0)$.

[*The standard symplectic form on* $\mathbb{R}^2$ *or* $\mathbb{R}^4$ *is denoted by* ω_0. *Recall that submanifolds* $N_1, N_2 \subset M$ *intersect cleanly if* $N_1 \cap N_2$ *is a smooth submanifold such that* $T_x(N_1 \cap N_2) = T_xN_1 \cap T_xN_2$ *for every* $x \in N_1 \cap N_2$.]

END OF PAPER