UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2021 12:00 pm to 3:00 pm

PAPER 144

MODEL THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let T be a complete theory in a countable language $\mathcal{L}$.

(a) Define the notion of a *prime model* of T.

(b) Define the notion of an *atomic model* of T.

(c) Suppose $\mathcal{M} \models T$ is countable and atomic. Prove that $\mathcal{M}$ is a prime model of T.

2 Let $\mathcal{L}$ be a first-order language and let T be a complete $\mathcal{L}$-theory with infinite models.

(a) State the Fundamental Theorem of Stability. [You do not need to give a proof.]

(b) Suppose that for any model $\mathcal{M} \models T$, any type in $S_1(M)$ is definable. Prove that T is stable.

(c) Given a model $\mathcal{M} \models T$ and a subset $X \subseteq M^n$, for some $n \geqslant 1$, we say that X is *definable in* $\mathcal{M}$ if there is an $\mathcal{L}_M$-formula $\varphi(x_1, \ldots, x_n)$ such that

$$X = \{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\}.$$

Assume T is stable and fix models $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \preceq \mathcal{N}$. Suppose $X \subseteq N^n$ is definable in $\mathcal{N}$. Prove that $X \cap M^n$ is definable in $\mathcal{M}$.

3 Let $\mathcal{L}$ be a countable first-order language and T be a complete $\mathcal{L}$-theory with infinite models. Fix a sub-language $\mathcal{L}_0 \subseteq \mathcal{L}$, and let T_0 be the set of all $\mathcal{L}_0$-sentences φ such that $T \models \varphi$.

(a) Show that T_0 is a complete $\mathcal{L}_0$-theory with infinite models.

(b) Suppose T is $\aleph_0$-categorical. Is T_0 necessarily $\aleph_0$-categorical (as an $\mathcal{L}_0$-theory)? [Justify your answer. You may use any of the characterisations of $\aleph_0$-categoricity proved in lecture, provided you state it clearly and correctly.]

(c) Suppose $\kappa > \aleph_0$ and T is κ-categorical. Is T_0 necessarily κ-categorical (as an $\mathcal{L}_0$-theory)? [Justify your answer.]

4 Let $\mathcal{L}$ be a first-order language and let T be an $\mathcal{L}$-theory.

(a) Define what it means for T to have *quantifier elimination.*

(b) Suppose $\mathcal{L} = \{<\}$ is the language of linear orders, and T is the theory of dense linear orders without endpoints. Prove that T has quantifier elimination. [You may use any of the characterisations of quantifier elimination proved in lecture, provided you state it clearly and correctly.]

END OF PAPER