UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 17 June, 2021 12:00 pm to 3:00 pm

PAPER 137

MODULAR FORMS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **THREE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $k \geqslant 4$ be an even integer.

(a) Write down the q-expansion $\sum_{n\geqslant 0} a_n(E_k)q^n$ of the normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k}.$$

Show that the coefficients $a_n(E_k)$ are rational numbers. Show that the coefficients are integers when $k = 4$ or $k = 6$.

(b) Prove that there exists a basis $f_0, \ldots, f_N$ of $M_k(\mathrm{SL}_2(\mathbb{Z}))$ such that for each $i = 0, \ldots, N$, the q-expansion of f_i is $f_i(\tau) = q^i + \sum_{n\geqslant N+1} a_n(f_i)q^n$ with $a_n(f_i) \in \mathbb{Z}$ for all n. [You may assume the formula for the number of zeroes of a modular form $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.]

(c) Write $a_1(E_k) = r/s$, where r, s are coprime integers. Show that if a prime number p divides s, then there exists a cuspidal modular form $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ such that $a_n(f) \in \mathbb{Z}$ and $a_n(f) \equiv \sigma_{k-1}(n) \bmod p$ for all $n \geqslant 1$, where $\sigma_{k-1}(n) = \sum_{d\in\mathbb{N}, d|n} d^{k-1}$.

(d) Prove the identity

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i)$$

for $n \in \mathbb{N}$.

[You may use that $\zeta(k)/\pi^k \in \mathbb{Q}$ and in particular that $\zeta(4) = \pi^4/(2 \times 9 \times 5)$, $\zeta(6) = \pi^6/(27 \times 5 \times 7)$, and $\zeta(8) = \pi^8/(2 \times 27 \times 25 \times 7)$.]

2 Let N, k be a positive integers.

(a) Define the congruence subgroup $\Gamma_1(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ and the spaces $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$.

(b) Prove that if $f \in S_k(\Gamma_1(N))$ then $f|_k[\alpha_N] \in S_k(\Gamma_1(N))$, where

$$\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

and $f|_k[\alpha_N](\tau) = \det(\alpha_N)^{k-1} f(\alpha_N\tau) j(\alpha_N, \tau)^{-k}$.

(c) Prove that if $f(\tau) = \sum_{n\geqslant 1} a_n(f)q^n \in S_k(\Gamma_1(N))$ and $L(f,s) = \sum_{n\geqslant 1} a_n(f)n^{-s}$, then $L(f,s)$ admits an analytic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(f,s) = i^k N^{1-k/2} \Lambda(f|_k[\alpha_N], k-s),$$

where $\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$. [You may assume that for any $f \in S_k(\Gamma_1(N))$, the series $L(f,s)$ is absolutely convergent in the region $\mathrm{Re}(s) > 1 + k/2$, and any relevant properties of the function $\Gamma(s)$.]

3 Let $\mathfrak{h} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ denote the complex upper half-plane.

(a) Let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Describe the topology and coordinate charts on the Riemann surface $X(\Gamma)$.

(b) Let $\pi : \mathfrak{h} \to \Gamma\backslash\mathfrak{h}$ be the quotient map and let f be a modular function of weight 0 and level Γ. Show that there is a unique morphism $\varphi_f : X(\Gamma) \to \widehat{\mathbb{C}}$ to the Riemann sphere such that $f = \varphi_f|_{\Gamma\backslash\mathfrak{h}} \circ \pi$.

(c) Write down a formula for the genus of $X(\Gamma)$, and use it to prove that the genus is 0 when $\Gamma = \Gamma_0(3)$.

(d) Show that $f(\tau) = \Delta(\tau)/\Delta(3\tau)$ is a modular function of weight 0 and level $\Gamma_0(3)$, where $\Delta(\tau) = (E_4(\tau)^3 - E_6(\tau)^2)/1728$. Decide, with proof, whether φ_f is an isomorphism. [You may assume any facts you need concerning the modular form $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.]

END OF PAPER