UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2021 12:00 pm to 3:00 pm

PAPER 136

LOCAL FIELDS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***FOUR*** *questions.*
There are ***FIVE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove a version of Hensel's Lemma.

(b) Find the number of solutions to the equation $X^3 + 5X - 12 = 0$ in $\mathbb{Q}_p$ for $p = 2, 3, 5$.

(c) Let p be an odd prime. Show that an element $x \in \mathbb{Q}_p^\times$ is a $(p-1)m^{\text{th}}$-power in $\mathbb{Q}_p$ for all $m \geqslant 1$ coprime to p, if and only if $x \in 1 + p\mathbb{Z}_p$.

2 (a) State and prove a classification of the (non-trivial) non-archimedean absolute values on $\mathbb{Q}$ up to equivalence. Explain with proof how this result generalises to number fields. [*Standard facts about Dedekind domains and DVR's may be used without justification*]

(b) Let $(K, |\cdot|)$ be a complete discretely valued field and L/K a finite extension. Show that if $|\cdot|$ extends to an absolute value $|\cdot|_L$ on L, then this extension is unique up to equivalence, and that L is complete with respect to $|\cdot|_L$.

Give an example to show how the uniqueness can fail if K is not assumed to be complete.

3 (a) Let K be a complete discretely valued field with $\text{char}(K) = p > 0$ and assume that its residue field k is perfect. Show that there is a unique ring homomorphism $[\cdot] : k \to \mathcal{O}_K$ lifting the identity on k and use this to show that $K \cong k((t))$. If in addition, K is locally compact, show that k is finite.

(b) Let p be an odd prime and let $K \cong \mathbb{F}_p((t))$.

(i) Let L/K be a finite Galois extension. Define the higher ramification groups $G_s(L/K)$ for $s \in \mathbb{Z}_{\geqslant -1}$.

(ii) Construct a finite Galois extension L/K satisfying both of the following conditions

$$G_{-1}(L/K)/G_0(L/K) \cong \mathbb{Z}/2\mathbb{Z},\ G_0(L/K) \cong \mathbb{Z}/(p^2-1)\mathbb{Z}.$$

4 (a) Let L/K be a finite extension of p-adic fields. Show that L/K is totally ramified if and only if $L = K(\alpha)$, where α is a root of an Eisenstein polynomial.

(b) Let ζ_{p^n} be a primitive p^n-th root of unity and let $L = \mathbb{Q}_p(\zeta_{p^n})$. Show that $L/\mathbb{Q}_p$ is a totally ramified Galois extension and that there is an isomorphism

$$\mathrm{Gal}(L/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times.$$

Deduce that there is an isomorphism $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$, where $\mathbb{Q}_p(\zeta_{p^\infty}) := \bigcup_{n=1}^{\infty} \mathbb{Q}_p(\zeta_{p^n})$.

(c) State the local Kronecker–Weber theorem for $\mathbb{Q}_p$ and describe how this can be used to define the Artin map

$$\mathrm{Art}_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \to W(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p).$$

5 (a) State the definition of a formal group law and show that if $F(X,Y) \in R[[X,Y]]$ is a formal group law over a ring R, there exists a power series $i(X) \in R[[X]]$ with $i(X) \equiv -X \mod X^2$ such that $F(X, i(X)) = 0$.

(b) Define the formal additive group $\widehat{\mathbb{G}}_a$ and the formal multiplicative group $\widehat{\mathbb{G}}_m$. Show that if R is a $\mathbb{Q}$-algebra, there is an isomorphism $\widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_m$.

(c) Recall that $\mathrm{End}_R(F)$ is a ring, hence there is a homomorphism $[\cdot]_F : \mathbb{Z} \to \mathrm{End}_R(F)$. Compute $[n]_{\widehat{\mathbb{G}}_a} \in R[[X]]$ and $[n]_{\widehat{\mathbb{G}}_m} \in R[[X]]$. Deduce that if R is a field of characteristic $p > 0$, there are no non-zero homomorphisms from $\widehat{\mathbb{G}}_a$ to $\widehat{\mathbb{G}}_m$.

END OF PAPER