UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 15 June, 2021 12:00 pm to 2:00 pm

PAPER 130

RAMSEY THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

We say that a finite subset X of $\mathbb{R}^n$ is *Ramsey* if for every k there exists a finite set S in $\mathbb{R}^m$, for some m, such that whenever S is k-coloured there is a monochromatic congruent copy of X. [Here as usual two sets are *congruent* if there is a bijection between them that preserves the Euclidean distance.]

Show that every set of size 2 is Ramsey, and also that every equilateral triangle is Ramsey.

By applying a product argument, show that if $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are Ramsey then the product set $X \times Y \subset \mathbb{R}^{n+m}$ is Ramsey. Deduce that every rectangle is Ramsey, and hence show that every right-angled triangle is Ramsey.

Show that the set $\{0, 1, 2\} \subset \mathbb{R}$ is not Ramsey. [Hint: use a colouring of $\mathbb{R}^m$ in which the colour of a point x depends only on its Euclidean norm $\|x\|$. You may find it useful to recall that for any two points x, y in $\mathbb{R}^m$ we have $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.]

2

State and prove the Hales-Jewett theorem.

State and prove Gallai's theorem.

Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) Whenever $\mathbb{N}^2$ is finitely coloured there is a monochromatic square whose side-length is even.

(ii) Whenever $\mathbb{N}^2$ is finitely coloured there is a monochromatic square whose side-length is odd.

(iii) Whenever $\mathbb{N}^2$ is finitely coloured there is a monochromatic square whose side-length is a power of 2.

[Here as usual 'square' means 'the four vertices of a square whose sides are parallel to the axes'.]

3

State Rado's theorem, making sure you define all terms that you use.

Prove Rado's theorem for the case of a single equation. [You may assume van der Waerden's theorem.]

Now let a_1, a_2, a_3, a_4 be non-zero rationals.

(i) If (a_1, a_2, a_3, a_4) is partition regular, does it follow that whenever $\mathbb{N}$ is finitely coloured there exist monochromatic x_1, x_2, x_3, x_4 such that $a_1x_1 + \cdots + a_4x_4 = 0$ and $x_1 + x_2 - x_3 - x_4 > 0$?

(ii) Suppose that whenever $\mathbb{N}$ is finitely coloured there exist monochromatic x_1, x_2, x_3, x_4 such that $a_1x_1 + \cdots + a_4x_4 > 0$, and also that whenever $\mathbb{N}$ is finitely coloured there exist monochromatic x_1, x_2, x_3, x_4 such that $a_1x_1 + \cdots + a_4x_4 < 0$. Does it follow that (a_1, a_2, a_3, a_4) is partition regular?

4

Define the terms *filter* and *ultrafilter*, and show that every filter extends to an ultrafilter.

Define the space $\beta\mathbb{N}$, and prove that it is compact and Hausdorff.

Let $\mathcal{S}$ be a family of subsets of $\mathbb{N}$. Prove that the following are equivalent:

(i) Whenever $\mathbb{N}$ is finitely coloured, some member of $\mathcal{S}$ is monochromatic.

(ii) There is an ultrafilter $\mathcal{U}$ such that every set $A \in \mathcal{U}$ contains some member of $\mathcal{S}$.

[Hint: which are the sets that *have* to belong to $\mathcal{U}$?]

Does there exist an ultrafilter $\mathcal{U}$ such that every member of $\mathcal{U}$ contains either a pair $\{x, 2x\}$ (for some x) or a pair $\{x, 3x\}$ (for some x)? Justify your answer.

END OF PAPER