UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 21 June, 2021 12:00 pm to 3:00 pm

PAPER 129

ADDITIVE COMBINATORICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question A is a fixed subset of $\mathbb{F}_2^n$ such that $|A+A| \leqslant K|A|$.

(a) Show that there exists $X \subseteq A$ such that for all $t \geqslant 1$

$$|X + tA| \leqslant K^t|X|.$$

(b) Show that there exists $Y \subseteq A + A$ such that $|Y| \leqslant 2K^2 - 1$ and

$$A + A - A - A \subseteq A - A + Y - Y.$$

[*Hint: Consider a sequence* $y_1, \ldots, y_m \in A + A$ *such that* $y_1 = 0$ *and for* $2 \leqslant j \leqslant m$

$$|(X + y_j) \setminus \cup_{1 \leqslant i < j} (X + y_i)| \geqslant \tfrac{1}{2}|X|,$$

where X *is the set from part (a).*]

(c) Deduce that A is contained in a coset of a subgroup of size at most $K2^{2K^2-2}|A|$.

(d) Give an example to show that the factor of $K2^{2K^2-2}$ in part (c) cannot be replaced by anything smaller than $K^{-1}2^K$.

2 This question takes place in some fixed finite abelian group G. You may assume without proof any standard results about Bohr sets proved in lectures, provided you state them clearly.

(a) Let $\Gamma \subset \widehat{G}$ and $0 \leqslant \rho \leqslant 2$. Define the *Bohr set* with frequency set Γ and width ρ.

(b) What does it mean for a Bohr set to be *regular*? Give an example (in a group of your choice) to show that not all Bohr sets are regular.

(c) Suppose that $|G|$ is odd and let B be a Bohr set of rank d. Show that B contains at least $(cd)^{-O(d)}|B|^2$ many three-term arithmetic progressions, where $c > 0$ is some absolute constant.

[*Hint: Consider those progressions* $x, x + y, x + 2y$ *where* $x + y \in B_\delta$ *for some appropriate* δ.]

(d) Suppose that $|G|$ is odd. State Bourgain's bound on the size of subsets of G without non-trivial three-term arithmetic progressions, and show how it follows from a suitable density increment result. [You do not need to prove the density increment result itself.]

3 This question takes place in $\mathbb{F}_p^n$ where $n \geqslant 1$ and p is some fixed prime.

(a) Let $m \geqslant 1$ and $X \geqslant 0$. For any $f : \mathbb{F}_p^n \to \mathbb{C}$, define the set of L^{2m} *almost-periods of f with error X*.

(b) Let $\epsilon > 0$ and $m \geqslant 1$. Show that if $A \subset \mathbb{F}_p^n$ has density $\alpha = |A|/p^n$ then the set of L^{2m} almost-periods of $1_A * 1_A$ with error $\epsilon|A|p^{n/2m}$ contains a subspace of codimension $O(m\epsilon^{-2})$.

[You may assume the Marcinkiewicz-Zygmund inequality and any standard properties of the Fourier transform.]

(c) Show that if $A \subset \mathbb{F}_p^n$ has density $\alpha = |A|/p^n$ then there is a subspace $V \leqslant \mathbb{F}_p^n$ of codimension $O(\epsilon^{-2}\log(1/\alpha))$ such that for any $x \in \mathbb{F}_p^n$ and $t \in V$,

$$|1_A * 1_A * 1_A(x+t) - 1_A * 1_A * 1_A(x)| \leqslant \epsilon|A|^2.$$

(d) Deduce that if $A \subset \mathbb{F}_p^n$ has density $\alpha = |A|/p^n$ then $A + A + A$ contains a translate of a subspace of codimension $O(\alpha^{-2}\log(1/\alpha))$.

4 All sets in this question are assumed to be finite. You may assume without proof any standard results about Freiman isomorphisms (including Ruzsa's modelling lemma) proved in lectures, provided you state them clearly.

(a) What does it mean for two sets A and B to be *Freiman* s*-isomorphic* for an integer $s \geqslant 1$?

(b) Show that for any $s \geqslant 1$ every $A \subset \mathbb{Z}^2$ is Freiman s-isomorphic to a subset of $\mathbb{Z}$.

(c) State the dense Bogolyubov-Ruzsa lemma for large subsets of cyclic groups of prime order, and use it to show that if $A \subset \mathbb{Z}$ has $|A + A| \leqslant K|A|$ then there is a proper generalised arithmetic progression P of rank $O_K(1)$ such that

$$|A \cap P| \gg_K |A| \gg_K |P|.$$

[You may assume Plünnecke's inequality and any covering lemmas proved in the course without proof.]

(d) State the Balog-Szemerédi-Gowers lemma.

(e) Let $k \geqslant 1$ and $\delta > 0$, and $f : \{1, \ldots, N\} \to \mathbb{Z}$. Show that if N is sufficiently large, depending only on k and δ, and there are at least δN^3 many $x, y, z, w \in \{1, \ldots, N\}$ such that

$$x + y = z + w \quad \text{and} \quad f(x) + f(y) = f(z) + f(w)$$

then there is a non-trivial arithmetic progression $P \subset \{1, \ldots, N\}$ of length k together with $a, b \in \mathbb{Q}$ such that

$$f(x) = ax + b \quad \text{for all} \quad x \in P.$$

[*Hint: Consider the graph* $\Gamma = \{(x, f(x)) : 1 \leqslant x \leqslant N\} \subset \mathbb{Z}^2$.]

[You may assume without proof Szemerédi's theorem: if $\delta > 0$ and $k \geqslant 1$ and Q is an arithmetic progression with $|Q|$ sufficiently large (depending on δ and k) then every subset of Q of size at least $\delta|Q|$ contains a non-trivial arithmetic progression of length k.]

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