MATHEMATICAL TRIPOS Part III

Friday, 18 June, 2021 12:00 pm to 3:00 pm

PAPER 127

HOMOTOPY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FIVE*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the homotopy groups of a space. What does it mean to say that a map is a weak homotopy equivalence? What does it mean to say that a map is n-connected? Define the term CW complex.

Prove that any space is weakly homotopy equivalent to a CW complex.

Give an example to show that a path-connected space X having $H_n(X;\mathbb{Z}) = 0$ and $\pi_n(X, x_0) = 0$ need not be weakly homotopy equivalent to a CW-complex with no n-cells.

2 What does it mean to say that a map $\pi : E \to B$ is a Serre fibration? Describe the long exact sequence on homotopy groups associated to a Serre fibration. [You do not need to prove that it is exact, but should carefully describe all the maps involved.]

Prove carefully that a fibre bundle is a Serre fibration.

Show that the inclusion $i : \mathbb{RP}^n \to \mathbb{CP}^n$ induces the trivial map on all homotopy groups, for all $n \geqslant 1$.

3 Let $\pi : E \to S^n$ be a Serre fibration with fibre $F := \pi^{-1}(*)$ over the basepoint $* \in S^n$, and suppose that $n \geqslant 2$. Construct, using the Serre spectral sequence of π, a homomorphism $\Delta : H_i(F) \to H_{i+n-1}(F)$ and a long exact sequence involving Δ and $H_*(E)$.

Calculate the integral homology groups and cohomology ring of the homotopy fibre of a map $f_m : S^n \to S^n$ of degree $m > 0$, for each $n \geqslant 2$. [Pay particular attention to the ring structure when $n = 2$.]

4 Calculate the rings $H^*(K(\mathbb{Z}/n, 1); \mathbb{Z})$ and $H^*(K(\mathbb{Z}/n, 1); \mathbb{Z}/p)$, for $n > 1$ an integer and p an odd prime number.

Let $u \in H^1(K(\mathbb{Z}/p, 1); \mathbb{Z}/p)$ denote a generator, and consider the map

$$f : K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1) \longrightarrow K(\mathbb{Z}/p, 2)$$

representing the cohomology class $u \times u$. Show that the homotopy fibre F of the map f is a $K(G, 1)$ and, by expressing F as the total space of a fibration and calculating part of its cohomology, prove that the group G is not abelian.

5 Let $\pi : E \to B$ be a Serre fibration over a path-connected space B, let $b_0 \in B$ and suppose that $F := \pi^{-1}(b_0)$ is also path-connected. Define what it means for a pair $(x, y) \in H^n(F) \times H^{n+1}(B)$ to be transgressive, and state the relationship between this notion and the Serre spectral sequence for $\pi : E \to B$.

State and prove Kudo's transgression theorem, describing the properties of Steenrod squares which you use.

Derive a formula for the action of the Steenrod operations on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$, describing the properties of Steenrod squares which you use.

Prove that for any space Y and any class $y \in H^n(Y; \mathbb{Z}/2)$ we have $\mathrm{Sq}^1\mathrm{Sq}^1(y) = 0$.

END OF PAPER