UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 22 June, 2021 12:00 pm to 3:00 pm

PAPER 123

ALGEBRAIC NUMBER THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let R be a Dedekind domain, K its field of fractions, and S the integral closure of R in a finite separable extension L/K. Explain what is meant by the *discriminant ideal* $\mathrm{disc}(S/R) \subset R$ and the *inverse different* $\mathfrak{D}_{S/R}^{-1} \subset L$. Show that the inverse different is a fractional ideal of S, whose inverse $\mathfrak{D}_{S/R}$ is an ideal of S, and that $N_{L/K}\mathfrak{D}_{S/R} = \mathrm{disc}(S/R)$.

Show that the class of $\mathrm{disc}(S/R)$ in the ideal class group $\mathrm{Cl}(R)$ is a square.

[You may assume without proof the compatibility of the different and discriminant with localisation.]

2 Let K be a number field. Define the ring of *adeles* $\mathbb{A}_K$ of K, and the topology on $\mathbb{A}_K$. Show that $K \subset \mathbb{A}_K$ is discrete and that the quotient $\mathbb{A}_K/K$ is compact. [You may use without proof the isomorphism $\mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K \simeq \mathbb{A}_K$.]

Define the group of *ideles* J_K of K, and the topology on J_K. Show that the inclusion $j \colon J_K \hookrightarrow \mathbb{A}_K$ is continuous. By considering the sequence $(x^{(i)})_{i\geqslant 1}$ in J_K, where for each $i \geqslant 1$,

$$x_v^{(i)} = \begin{cases} p_i & \text{if } v|p_i, \text{ the } i\text{-th prime number} \\ 1 & \text{otherwise} \end{cases}$$

show that j is not a homeomorphism onto its image.

3 Let F be a finite extension of $\mathbb{Q}_p$. What is the *Schwartz space* $\mathcal{S}(F)$? Define the *Fourier transform* $\hat{f}$ of a function $f \in \mathcal{S}(F)$, and compute $\hat{f}$ when f is the characteristic function of $\mathcal{O}_F$.

Show that if $f \in \mathcal{S}(F)$, a, $b \in F$ with $a \neq 0$ and $g(x) = f(ax+b)$, then $\hat{g}(y) = \psi(-by/a)\,|a|^{-1}\,\hat{f}(y/a)$. Deduce that Fourier transform maps $\mathcal{S}(F)$ to itself.

4 (i) Show that there is no Galois extension $L/\mathbb{Q}$ with Galois group S_3 which is unramified outside of 7.

(ii) Compute the ray class group $Cl_{\mathfrak{m}}(K)$, where $K = \mathbb{Q}(\sqrt{-6})$ and $\mathfrak{m} = 2(v_2)+2(v_3)$ where for $p \in \{2,3\}$, v_p is the unique place dividing p. [You may use without proof that K has class number 2.]

END OF PAPER