UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 25 June, 2021 12:00 pm to 3:00 pm

## PAPER 120

## COMPUTABILITY AND LOGIC

***Before you begin please read these instructions carefully***

*Candidates have THREE HOURS to complete the written examination.*

*Attempt no more than **FOUR** questions.*
*There are **SIX** questions in total.*
*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

State and prove Knaster-Tarski. State and prove Myhill-Nerode.

**2**

Give a proof of the following statement. If $f$ is a function $\mathbb{N}^k \to \mathbb{N}$ declared by primitive recursion, then there is a formula $\phi(y, x_1 \ldots x_k, \vec{z})$ in the language with 0, 1, +, $\times$, $<$ and $=$ ("the language of ordered rings"), which contains no unrestricted quantifiers, and is such that

$$y = f(x_1 \ldots x_k) \text{ iff } (\exists \vec{z})\phi(y, \vec{x}, \vec{z}).$$

**3**

Explain how every partial computable function $\mathbb{N}^k \to \mathbb{N}$ can be captured by $\lambda$-terms acting on Church numerals. You should explain how primitive recursion and minimisation can be captured. You need not prove the "completeness theorem" to the effect that these two devices capture all machine-computable functions.

**4**

(a)

State and justify a principle of structural induction over primitive recursive functions, and use it to prove that every primitive recursive function is total.

(b)

Give a definition of the natural numbers that does not involve quantifying over infinite sets. Prove its equivalence with the usual definition.

(c)

Show that for every theory in a first order-language with a semidecidable set of axioms there is an equivalent independent axiomatisation that is decidable.

**5**

Show that the set of gnumbers of total computable functions is not recursively axiomatisable, but that, for any recursively axiomatisable theory $T$ of arithmetic, the set of gnumbers of computable functions that $T$ proves to be total is recursively axiomatisable. Hence or otherwise prove that for every sound recursively axiomatised theory $T$ of arithmetic we can obtain a total computable function $f_T$ whose totality is not provable in $T$.

**6**

(a)

Prove the following weak version of a theorem of Tennenbaum:

**Theorem 1.** *If* $\mathfrak{M}$ *is a nonstandard model of true arithmetic with carrier set* $\mathbb{N}$ *then the graphs of* $+$ *and* $\times$ *in* $\mathfrak{M}$ *cannot both be decidable.*

(b)

Is the set of first-order sentences true in all finite structures recursively enumerable? Justify your answer.

## END OF PAPER