UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2021 12:00 pm to 3:00 pm

PAPER 119

CATEGORY THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***FOUR*** *questions.*
There are ***SIX*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the Yoneda Lemma, and use it to prove that if $\mathcal{C}$ is a small category then every object of the functor category $[\mathcal{C}, \mathbf{Set}]$ is an epimorphic image of a projective object. [You may assume the result that epimorphisms in $[\mathcal{C}, \mathbf{Set}]$ are pointwise surjective natural transformations.]

A functor $F\colon \mathcal{C} \to \mathbf{Set}$ is called a *monofunctor* if Ff is injective for every morphism f of $\mathcal{C}$. Show that the following conditions on a small category $\mathcal{C}$ are equivalent:

(i) Every morphism of $\mathcal{C}$ is monic.

(ii) Every representable functor $\mathcal{C} \to \mathbf{Set}$ is a monofunctor.

(iii) Every functor $\mathcal{C} \to \mathbf{Set}$ is an epimorphic image of a monofunctor.

Under what conditions on $\mathcal{C}$ is *every* functor $\mathcal{C} \to \mathbf{Set}$ a monofunctor? Justify your answer.

2 Given an adjunction $(F\colon \mathcal{D} \to \mathcal{C} \dashv G\colon \mathcal{C} \to \mathcal{D})$, define the *unit* and *counit* of the adjunction, and explain briefly how the adjunction may be recovered from its unit and counit.

Given an adjunction $(F \dashv G)$, show that the following conditions are equivalent:

(i) F is full and faithful.

(ii) The unit of the adjunction is an isomorphism.

(iii) There exists an isomorphism between GF and the identity functor on $\mathcal{D}$.

Hence or otherwise show that if G has a right adjoint H as well as a left adjoint F, then H is full and faithful if and only if F is.

Now suppose given a double adjunction $(F \dashv G \dashv H)$ where G is full and faithful; let η and ϵ denote the unit and counit of $(F \dashv G)$, and α and β those of $(G \dashv H)$. Show that the composites $(\alpha_F)^{-1}(H\eta)\colon H \to HGF \to F$ and $(F\beta)(\epsilon_H)^{-1}\colon H \to FGH \to F$ are equal [*hint: consider their images under* G]. Show also that this natural transformation $\theta\colon H \to F$ is pointwise monic (respectively epic) if and only if F (respectively H) acts faithfully on morphisms whose domains (respectively codomains) are in the image of G. [*Given a morphism* $f\colon B \to HA$, *consider* $G(\theta_A f)$.]

3 Explain carefully what is meant by a (or the) *colimit* of a diagram $D\colon J \to \mathcal{C}$.

If I and J are small categories, a functor $F\colon I \to J$ is said to be *final* if, for every $j \in \mathrm{ob}\ J$, the category $(j \downarrow F)$ is (nonempty and) connected. Show that if $F\colon I \to J$ is final then, for any $D\colon J \to \mathcal{C}$, each cone under DF extends uniquely to a cone under D, and deduce that if $\mathcal{C}$ has colimits of shape I then it has colimits of shape J.

A small category J is said to be *sifted* if it is connected and the diagonal functor $J \to J \times J$ is final. Show that if J is sifted then the functor $\mathrm{colim}_J\colon [J, \mathbf{Set}] \to \mathbf{Set}$ preserves finite products. [*Hint: Given diagrams* $D, E : J \rightrightarrows \mathbf{Set}$*, show that* $\mathrm{colim}_J D \times \mathrm{colim}_J E$ *may be expressed as the colimit of a suitable diagram of shape* $J \times J$*. You may assume the result that functors of the form* $(-) \times A\colon \mathbf{Set} \to \mathbf{Set}$ *have right adjoints.*]

4 Explain what is meant by a *monad* $\mathbb{T}$ on a category $\mathcal{C}$, and define the Eilenberg–Moore category $\mathcal{C}^{\mathbb{T}}$ and the Kleisli category $\mathcal{C}_{\mathbb{T}}$ associated with such a monad. [You need not verify that they are categories.]

Let $\mathbb{T} = (T, \eta.\mu)$ be a monad on $\mathcal{C}$, and let $\mathcal{D}$ be an arbitrary category. Show that the functors $T_*\colon [\mathcal{D}, \mathcal{C}] \to [\mathcal{D}, \mathcal{C}]$ and $T^*\colon [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$, defined on objects by $F \mapsto TF$ and $G \mapsto GT$ respectively, both carry monad structures, and that the Eilenberg–Moore categories of the resulting monads $\mathbb{T}_*$ and $\mathbb{T}^*$ are respectively equivalent to $[\mathcal{D}, \mathcal{C}^{\mathbb{T}}]$ and to $[\mathcal{C}_{\mathbb{T}}, \mathcal{D}]$. [*Hint for the latter: Show that* $\mathbb{T}^*$*-algebra structures on a functor* $G\colon \mathcal{C} \to \mathcal{D}$ *are equivalent to factorizations of* G *through* $F_{\mathbb{T}}\colon \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$*.*]

5 Define a *regular category*, and prove that in a regular category regular epimorphisms coincide with covers (that is, strong epimorphisms).

Let $\mathcal{C}$ be a category with finite limits, and let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ for which the reflector L (the left adjoint of the inclusion) preserves finite limits. For any object A of $\mathcal{C}$, we define a unary operation c on (isomorphism classes of) subobjects of A by taking $c(A' \rightarrowtail A)$ to be the left edge of the pullback square

$$\begin{array}{ccc} c(A') & \longrightarrow & LA' \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow{\ \eta_A\ } & LA \end{array}$$

where η is the unit of the adjunction. Show that c is a closure operation on $\mathrm{Sub}_{\mathcal{C}}(A)$ (that is, it is order-preserving and satisfies $A' \leqslant c(A') \cong c(c(A'))$ for any $A' \rightarrowtail A$), and that closure commutes with pullback along any morphism of $\mathcal{C}$. Show also that if $A \in \mathrm{ob}\ \mathcal{D}$, then a subobject $A' \rightarrowtail A$ belongs to $\mathcal{D}$ if and only if it is closed (i.e. isomorphic to its closure). Deduce that if $\mathcal{C}$ is regular then so is $\mathcal{D}$.

[*You may assume that* $\mathcal{D}$ *coincides with the full subcategory* $\mathrm{Fix}(L)$ *on objects* A *for which* η_A *is an isomorphism.*]

6 Explain what is meant by a *normal monomorphism* in a pointed category. If $\mathcal{C}$ is pointed and has kernels and cokernels, show that a monomorphism in $\mathcal{C}$ is normal if and only if it is the kernel of its own cokernel.

Define an *abelian category*, and show that an abelian category is regular.

By a *chain complex* $C_\bullet$ in an abelian category $\mathcal{A}$, we mean an infinite sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$$

of objects and morphisms such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Show that chain complexes in $\mathcal{A}$ may be identified with additive functors $\mathcal{Z} \to \mathcal{A}$, where $\mathcal{Z}$ is a suitable additive category with ob $\mathcal{Z} = \mathbb{Z}$ and

$$\begin{aligned} \mathcal{Z}(n,p) &= \mathbb{Z} \quad \text{if } p = n \text{ or } n-1 \\ &= \{0\} \quad \text{otherwise.} \end{aligned}$$

Given a chain complex $C_\bullet$, we write $Z_n(C_\bullet) \rightarrowtail C_n$ for the kernel of $d_n \colon C_n \to C_{n-1}$, $B_n(C_\bullet) \rightarrowtail C_n$ for the image of d_{n+1}, and $Z_n(C_\bullet) \twoheadrightarrow H_n(C_\bullet)$ for the cokernel of $B_n(C_\bullet) \rightarrowtail Z_n(C_\bullet)$. Show that the definition of H_n is self-dual (i.e., that if we regard $C_\bullet$ as a chain complex in $\mathcal{A}^{\mathrm{op}}$, and follow the construction just given, we arrive at the same object).

END OF PAPER