UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 9 June, 2021 12:00 pm to 3:00 pm

PAPER 115

DIFFERENTIAL GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a smooth n-manifold.

(a) Define what it means for X to be *orientable*.

Let M be the Möbius band, defined as the quotient

$$\big(\mathbb{R} \times (-1,1)\big)/\mathbb{Z},$$

where $k \in \mathbb{Z}$ acts via $(x,t) \mapsto (x + 2\pi k, (-1)^k t)$.

(b) By computing a suitable mod-2 Euler number, show that M is non-orientable.

Back in the general case, let $Y \subset X$ be a properly embedded codimension-1 submanifold.

(c) Construct a real line bundle $\mathcal{L} \to X$, and a section s of $\mathcal{L}$ which is transverse to the zero section, such that Y is given by the vanishing of s.

[*Hint: Cover X with coordinate patches U_α on which Y is given by the vanishing of a coordinate function f_α, and define $\mathcal{L}$ by using the ratios of the f_α on overlaps to define the transition functions.*]

[You may use the following fact without proof: if f is a smooth function on an open set $U \subset \mathbb{R}^n$, with standard coordinates $x_1, \ldots, x_n$, and if f vanishes when $x_1 = 0$, then the function h on U defined by

$$h(\mathsf{x}) = \begin{cases} f(\mathsf{x})/x_1 & \text{if } x_1 \neq 0 \\ \frac{\partial f}{\partial x_1}(\mathsf{x}) & \text{if } x_1 = 0 \end{cases}$$

is smooth.]

(d) State a classification result for line bundles on $\mathbb{R}^n$ and deduce that any properly embedded codimension-1 submanifold $Y \subset \mathbb{R}^n$ is orientable.

(e) Can the Möbius band M be (necessarily non-properly) embedded in $\mathbb{R}^3$? Either draw or write down an embedding, or justify briefly why no such embedding can exist.

2 Consider $\mathbb{CP}^1$ with homogeneous coordinates $[z_0 : z_1]$. Let U_0 and U_1 denote the open sets defined by $z_0 \neq 0$ and $z_1 \neq 0$ respectively, and let $z = re^{i\theta}$ be the (complex) local coordinate on U_0 given by z_1/z_0. We think of S^1 as the unit circle $\{r = 1\}$ in U_0, locally parametrised by θ and oriented in the direction of increasing θ.

(a) Show that the map $H^1_{\mathrm{dR}}(S^1) \to \mathbb{R}$ given by

$$[\alpha] \mapsto \int_{S^1} \alpha \qquad (\star)$$

is well-defined, and an isomorphism.

(b) Deduce that $(\star)$ also defines as isomorphism $H^1_{\mathrm{dR}}(U_0 \cap U_1) \to \mathbb{R}$. [You may assume standard invariance properties of de Rham cohomology as long as they are stated clearly.]

Let $\mathcal{L}$ be a complex line bundle on $\mathbb{CP}^1$.

(c) State briefly why $\mathcal{L}$ can be trivialised over U_0 and U_1.

Fix trivialisations Φ_0 and Φ_1 over U_0 and U_1, and let

$$f : U_0 \cap U_1 \to \mathbb{C}^*$$

be the transition function g_{10}. Let β be the complex-valued 1-form on $U_0 \cap U_1$ given by $f^{-1}\mathrm{d}f$.

(d) Show that for a smooth path $\gamma : \mathbb{R} \to U_0 \cap U_1$ the function

$$F(t) = f(\gamma(t))e^{-\int_0^t \gamma^*\beta}$$

is independent of t.

(e) Deduce that $\int_{S^1} \beta = 2\pi i k$ for some integer k.

Let $h = z^{-k} f$ and $\alpha = h^{-1}\mathrm{d}h$.

(f) Show that $\alpha = \mathrm{d}\varphi$ for some smooth complex-valued function φ on $U_0 \cap U_1$.

(g) Show that by adding a constant to φ if necessary we may arrange that $h = e^{\varphi}$.

(h) Construct smooth maps $\psi_j : U_j \to \mathbb{C}^*$ such that

$$\frac{\psi_1}{\psi_0} f = z^k.$$

[*Hint: take a partition of unity* ρ_0, ρ_1 *subordinate to the cover* $\{U_0, U_1\}$ *and write* φ *as* $\rho_0\varphi + \rho_1\varphi$.]

(i) Deduce that $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(-k)$.

3 Let G be a Lie group and $\pi : P \to B$ a principal G-bundle.

(a) Define a *connection* $\mathcal{A}$ on P, and its *curvature* $\mathcal{F}$.

Let $P = \{(\mathsf{x}, \mathsf{y}) \in S^n \times S^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \langle \mathsf{x}, \mathsf{y} \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{n+1}$.

(b) Show that P is a submanifold of $S^n \times S^n$, and describe $T_{(\mathsf{x},\mathsf{y})}P$ as a subspace of $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$.

Let $G = \mathrm{O}(2)$ and define a right G-action on P by

$$(\mathsf{x}, \mathsf{y}) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\mathsf{x} + c\mathsf{y}, b\mathsf{x} + d\mathsf{y}).$$

(c) Explain briefly why the quotient map $\pi : P \to B = P/G$ is a principal G-bundle.

(d) Show that there is a unique connection $\mathcal{A}$ on the G-bundle P such that a path (γ_1, γ_2) in $P \subset S^n \times S^n$ is horizontal if and only if $\langle \gamma_i', \gamma_j \rangle = 0$ for all i and j. (Here γ_i' denotes the derivative of γ_i with respect to its parameter.)

(e) Show that for this connection we have

$$\mathcal{F}\big((\mathsf{u}_1, \mathsf{v}_1), (\mathsf{u}_2, \mathsf{v}_2)\big) = \big(\langle \mathsf{u}_2, \mathsf{v}_1 \rangle - \langle \mathsf{u}_1, \mathsf{v}_2 \rangle\big) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for all (x, y) in P and all $(\mathsf{u}_1, \mathsf{v}_1)$ and $(\mathsf{u}_2, \mathsf{v}_2)$ in $T_{(\mathsf{x},\mathsf{y})}P \subset \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$.

4 Let G be a Lie group.

(a) Define the *Lie algebra* $\mathfrak{g}$ as a vector space and show that the map $\xi \mapsto \mathsf{l}_\xi$ gives a linear isomorphism

$$\mathfrak{g} \to \{\text{left-invariant vector fields on } G\},$$

where l_ξ is a vector field that you should define explicitly.

(b) Carefully define the *Lie bracket* on $\mathfrak{g}$. You may assume the existence and properties of the Lie bracket of vector fields as long as you state them clearly.

Fix ξ in $\mathfrak{g}$ and define the curve $\gamma_\xi : \mathbb{R} \to G$ by $\gamma_\xi(t) = \exp(t\xi)$.

(c) Write down the defining ODE satisfied by γ_ξ.

Let (X, g) be a Riemannian manifold, and let ∇ denote the covariant derivative associated to the Levi-Civita connection.

(d) Show that $\nabla_{\mathsf{u}}\mathsf{v} - \nabla_{\mathsf{v}}\mathsf{u} = [\mathsf{u}, \mathsf{v}]$ for all vector fields u and v on X. [You may use standard properties of the Christoffel symbols $\Gamma^i{}_{jk}$ as long as you clearly state which intrinsic properties of the connection they correspond to.]

(e) Show that for all vector fields u, v, and w on X we have

$$\nabla_{\mathsf{u}}(g(\mathsf{v},\mathsf{w})) + \nabla_{\mathsf{v}}(g(\mathsf{u},\mathsf{w})) - \nabla_{\mathsf{w}}(g(\mathsf{u},\mathsf{v})) = 2g(\nabla_{\mathsf{u}}\mathsf{v},\mathsf{w}) - g([\mathsf{u},\mathsf{v}],\mathsf{w}) + g(\mathsf{v},[\mathsf{u},\mathsf{w}]) + g(\mathsf{u},[\mathsf{v},\mathsf{w}]).$$

Now suppose $X = G$ and that the metric g is left-invariant.

(f) Show that the curve γ_ξ defined above is a geodesic if and only if $\langle \xi, [\xi, \eta] \rangle = 0$ for all η in $\mathfrak{g}$, where $\langle \cdot, \cdot \rangle$ is the restriction of g to $\mathfrak{g}$.

END OF PAPER