UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2021 12:00 pm to 3:00 pm

PAPER 114

ALGEBRAIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be obtained from the two-torus $S^1 \times S^1$ by first attaching a two-cell along the circle $\{\theta_0\} \times S^1$ by a degree one homeomorphism, and by then collapsing the circle $S^1 \times \{\theta_1\}$ to a point. Exhibit a cell structure on X, and hence or otherwise compute its homology groups.

State the excision theorem.

Compute the groups $H_*(X, X\backslash\{x\}; \mathbb{Z})$ for each $x \in X$, and deduce that any homeomorphism $\phi : X \to X$ must preserve the subset (which is the image in X of) $[\{\theta_0\} \times S^1]$.

Need every homeomorphism of X preserve the point $p = [(\theta_0, \theta_1)]$? Briefly justify your answer.

2 For a space X, let ΣX denote the suspension of X. Compute the homology groups of ΣX in terms of those of X. Assuming that X is homotopy equivalent to a finite cell complex, express the Euler characteristic $\chi(\Sigma X)$ in terms of $\chi(X)$. Hence, or otherwise, prove that if $j \geqslant 0$ then $\Sigma^j(\mathbb{CP}^2)$ is not homotopy equivalent to $A \times A$, for any finite cell complex A.

Explain how to define a *degree* for homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$, and show that your definition has the properties that

- $deg(f) \in \pm 1$;
- $deg(f \circ g) = deg(f) \cdot deg(g)$;
- $deg(A) = \text{sign}(det(A))$ if $A \in GL(n, \mathbb{R})$.

For a space Y, let $\sigma : Y^4 \to Y^4$ be the cyclic permutation

$$\sigma(y_1, y_2, y_3, y_4) = (y_4, y_1, y_2, y_3).$$

By considering σ^2, or otherwise, prove that there is no topological space Y for which $Y \times Y$ is homeomorphic to $\mathbb{R}^{2n+1}$.

3 Let X be a topological space and suppose $m \in \mathbb{Z}$ satisfies $m > 1$. Define the Bockstein homomorphisms

$$\tilde{\beta} : H^i(X;\mathbb{Z}/m) \to H^{i+1}(X;\mathbb{Z})$$

and

$$\beta : H^i(X;\mathbb{Z}/m) \longrightarrow H^{i+1}(X;\mathbb{Z}/m)$$

and explain how they are related. Show that

$$\beta(x \smile y) = \beta(x) \smile y + (-1)^{|x|}\, x \smile \beta(y)$$

where $\smile$ denotes cup-product and $|\cdot|$ denotes the cohomological degree.

Now suppose $p > 2$ is prime. Stating clearly any other results you use, prove that there is no closed five-dimensional manifold M which satisfies

$$H^i(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0,5 \\ \mathbb{Z}/p & i = 3 \\ 0 & \text{otherwise.} \end{cases}$$

[*Hint: it may help to prove that* $\beta : H^2(M;\mathbb{Z}/p) \to H^3(M;\mathbb{Z}/p)$ *would be an isomorphism, and to consider* $\beta(x \smile x)$ *for* $x \in H^2(M;\mathbb{Z}/p)$*.*]

4 Let $E \to X$ be an oriented vector bundle of real rank k. Define the *Euler class* of E.

Let $k \geqslant 1$. Let $\mathcal{E}_k(X) \subset H^k(X;\mathbb{Z})$ be the set of Euler classes of oriented rank k vector bundles over X. Stating carefully any general results you use, prove that $\mathcal{E}_{2k}(S^{2k})$ contains all even integers. Deduce that if M is a closed $2k$-dimensional manifold then $2H^{2k}(M;\mathbb{Z}) \subset \mathcal{E}_{2k}(M)$. Give an example of a pair $\{M, k\}$ where this inclusion is strict.

If $\dim_{\mathbb{R}}(M) = 2k+1$, is there always an inclusion $2H^{2k+1}(M;\mathbb{Z}) \subset \mathcal{E}_{2k+1}(M)$? Briefly justify your answer.

Prove that (for $k, l \geqslant 1$) cup-product

$$H^k(X;\mathbb{Z}) \otimes H^l(X;\mathbb{Z}) \longrightarrow H^{k+l}(X;\mathbb{Z}) \tag{1}$$

induces a map

$$\mathcal{E}_k(X) \otimes \mathcal{E}_l(X) \longrightarrow \mathcal{E}_{k+l}(X). \tag{2}$$

Give examples to show that (2) need not be injective or surjective, briefly justifying your answers.

END OF PAPER