UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2021 12:00 pm to 3:00 pm

## PAPER 113

## ALGEBRAIC GEOMETRY

***Before you begin please read these instructions carefully***

*Candidates have THREE HOURS to complete the written examination.*

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) Let $S$ and $T$ be graded rings and $\varphi : S \to T$ be a graded ring homomorphism, i.e., $\varphi$ preserves degrees. Let

$$U := \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \varphi(S_+)\},$$

where $S_+ \subseteq S$ is the irrelevant ideal.

Show that $U$ is an open subset of $\operatorname{Proj} T$.

Show that $\varphi$ determines a morphism

$$f : U \to \operatorname{Proj} S.$$

(b) Now suppose that in part (a), $\varphi$ induces an isomorphism $\varphi_d : S_d \to T_d$ for all $d \geqslant d_0$, for some non-negative integer $d_0$. Show that $U = \operatorname{Proj} T$ and that the induced morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

**2** Let $k$ be a field, and let $X = \operatorname{Spec} k[x, y, z, w]/(xy - zw) \subseteq \mathbb{A}^4_k$.

(a) Show that $D = V(x, z)$ is a prime (Weil) divisor on $X$ and that $\operatorname{Cl} X \cong \mathbb{Z}$ is generated by the divisor class of $D$. [Note: you may assume that $X$ satisfies the hypotheses for the definition of class group.]

(b) Let $Y = X \setminus \{O\}$ where $O$ denotes the origin in $\mathbb{A}^4_k$. Show that $D_Y := D \cap Y$ defines a Cartier divisor on $Y$, and describe the line bundle $\mathcal{O}_Y(D_Y)$ via transition maps on a suitable open cover of $Y$.

(c) Show that there is a morphism $f : Y \to \mathbb{P}^1_k$ such that $f^*\mathcal{O}_{\mathbb{P}^1_k}(1) \cong \mathcal{O}_Y(D_Y)$.

**3** (a) Let $A$ and $B$ be rings, $\varphi : B \to A$ a ring homomorphism inducing a morphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$. Let $M$ be an $A$-module, and let $M_B$ denote the $B$-module with underlying group $M$ and multiplication $b \cdot m = \varphi(b)m$ for $b \in B$, $m \in M$. Show an equality of sheaves

$$f_*(\widetilde{M}) = \widetilde{M_B}.$$

(b) Let $f : X \to Y$ be a morphism of schemes with $Y$ affine and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Suppose further that $X$ is Noetherian. By choosing a suitable affine cover $\{U_i\}$ of $X$, a suitable affine cover $\{U_{ijk}\}$ of $U_i \cap U_j$, and a suitable morphism of sheaves

$$\bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}}),$$

show that $f_*\mathcal{F}$ is a quasi-coherent sheaf on $Y$.

**4** Let $X$ be a topological space.

(a) If $\mathcal{F}$ is a sheaf of abelian groups on $X$ and $s \in \Gamma(X, \mathcal{F})$, we write

$$\operatorname{Supp} s = \{p \in X \mid 0 \neq s_p \in \mathcal{F}_p\},$$

where $s_p$ denotes the germ of $s$ at $p$. Show that $\operatorname{Supp} s$ is a closed subset of $X$.

(b) Let $Z \subseteq X$ be a closed subset of $X$. We define

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \mid \operatorname{Supp} s \subseteq Z\}.$$

Show that the sequence

$$0 \to \Gamma_Z(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X \setminus Z, \mathcal{F}) \tag{1}$$

is exact, with surjectivity on the right if $\mathcal{F}$ is flasque.

Show that if

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \tag{2}$$

is an exact sequence of sheaves of abelian groups on $X$, then

$$0 \to \Gamma_Z(X, \mathcal{F}_1) \to \Gamma_Z(X, \mathcal{F}_2) \to \Gamma_Z(X, \mathcal{F}_3) \tag{3}$$

is exact. Show furthermore that if $\mathcal{F}_1$ is flasque, then surjectivity also holds on the right.

[Note: you may use the corresponding surjectivity statement for the functor $\Gamma(X, \cdot)$ when $\mathcal{F}_1$ is flasque without proof.]

(c) If $Z \subseteq X$ is a closed subset as in (b), we denote by $H^i_Z(X, \cdot)$, $i \geqslant 0$, the right derived functors of $\Gamma_Z(X, \cdot)$, whose existence you may assume without proof.

Now let $X = \operatorname{Spec} k[x, y]$, and let $Z = V(x, y)$. Calculate $H^i_Z(X, \mathcal{O}_X)$.

You may use the following properties of the functors $H^i_Z$ without proof:

1. $H^0_Z(X, \mathcal{F}) = \Gamma_Z(X, \mathcal{F})$;
2. Given an exact sequence of sheaves (2), there is a long exact sequence
$$\cdots \to H^{i-1}_Z(X, \mathcal{F}_3) \to H^i_Z(X, \mathcal{F}_1) \to H^i_Z(X, \mathcal{F}_2) \to H^i_Z(X, \mathcal{F}_3) \to H^{i+1}_Z(X, \mathcal{F}_1) \to \cdots .$$
3. For a sheaf $\mathcal{F}$ on $X$, the exact seqeuence of (1) extends to a long exact sequence
$$\cdots \to H^{i-1}(X \backslash Z, \mathcal{F}) \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(X \backslash Z, \mathcal{F}) \to H^{i+1}_Z(X, \mathcal{F}) \to \cdots .$$

## END OF PAPER