UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 23 June, 2021 12:00 pm to 3:00 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt **ALL** questions.*
*There are **THREE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Find a majorant function $g(x)$ for the function $f(x) = \cos(x_1 + x_2 + \cdots + x_n)$ for every $x \in \mathbb{R}^n$. [Express your answer in analytic form and not only as a power series].

(b) Consider the second order partial differential equation

$$u_t - (x^2 - 1)u_{xx} = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^2 . \qquad (*)$$

(i) Find the characteristic curves to this equation.

(ii) Let Σ be the analytic curve: $x = \sin s,\ t = s^2$ for $s \in \mathbb{R}$. Find all the values of $s \in \mathbb{R}$ for which the corresponding points on Σ are characteristic points for equation $(*)$.

(iii) Consider equation $(*)$ with the data

$$u(0,t) = \cos t\,, \quad u_x(0,t) = 0\,. \qquad (**)$$

For which $t \in \mathbb{R}$ does the above initial value problem, $(*)$ and $(**)$, have an analytic solution in the neighbourhood of $(0,t)$? [Justify your answer, carefully stating any theorems you use.]

2

(a) Let $U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1/4\}$.

(i) Show that there exists a function $u \in H^1(U)$ such that $u \notin L^\infty(U)$.

(ii) Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(U)$. Show that there is a subsequence $\{u_{n_k}\}_{k=1}^\infty$ and $u \in H^1(U)$ such that $\lim_{k\to\infty} \|u_{n_k} - u\|_{L^p(U)} = 0$, for each $p \in [1,\infty)$. [The same subsequence for all $p \in [1,\infty)$.]

(b) Let $U \subset \mathbb{R}^3$ be an open bounded set.

(i) Show that there is a constant $c > 0$ such that for each $v \in H^1(U)$ we have:

$$\|v\|_{L^3(U)} \leqslant c\|v\|_{L^2(U)}^{1/2}\|v\|_{H^1(U)}^{1/2}.$$

(ii) Let $v, w \in H^1(U)$ be given. Define the map $\Phi : H^1(U) \to \mathbb{R}$:

$$\Phi(u) = \int_U \Big(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\Big) vw \, dxdydz\,, \quad \text{for every} \quad u \in H^1(U)\,.$$

Show that Φ is a linear bounded functional.

(iii) Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(U)$. Show that there is a subsequence $\{u_{n_k}\}_{k=1}^\infty$ and $u \in H^1(U)$ such that

$$\lim_{k\to\infty} \int_U \Big(\frac{\partial u_{n_k}}{\partial x} + \frac{\partial u_{n_k}}{\partial y} + \frac{\partial u_{n_k}}{\partial z}\Big) u_{n_k} w \, dxdydz = \int_U \Big(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\Big) uw \, dxdydz\,,$$

for every $w \in H^1(U)$.

3

Let $U \subset \mathbb{R}^3$ be an open bounded set with $\partial U \in C^2$ and $f \in L^2(U)$. Consider the nonlinear elliptic boundary value problem:

$$\begin{aligned} -\Delta u - |Du|^2 u = f \quad &\text{in} \quad U \\ u = 0 \quad &\text{on} \quad \partial U\,. \end{aligned} \tag{†}$$

(i) Use elliptic theory and regularity [without proof] to show that the map Φ introduced below is well defined.

$\Phi : H^2(U) \cap H_0^1(U) \to H^2(U) \cap H_0^1(U)$, where for every $w \in H^2(U) \cap H_0^1(U)$ $v = \Phi(w)$ is defined by solving the elliptic boundary value problem:

$$\begin{aligned} -\Delta v - |Dw|^2 w = f \quad &\text{in} \quad U \\ v = 0 \quad &\text{on} \quad \partial U\,. \end{aligned}$$

(ii) Show that there exist $r, R > 0$ (small enough) such that Φ is a contraction map from the closed ball $B_R(0) = \{w \in H^2(U) \cap H_0^1(U) : \|w\|_{H^2(U)} \leqslant R\}$ into itself, provided $\|f\|_{L^2(U)} \leqslant r$. Hence deduce that the map Φ has a fixed point and conclude that the boundary value problem (†) has a solution.

[State carefully the theorems that you use without proof].

END OF PAPER