MATHEMATICAL TRIPOS Part III

Friday, 11 June, 2021 12:00 pm to 3:00 pm

PAPER 102

FINITE DIMENSIONAL LIE AND ASSOCIATIVE ALGEBRAS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **FOUR** questions.*
*There are **FIVE** questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let V be a finite dimensional complex vector space. Describe the Lie algebra structure on the space $\mathrm{End}(V)$ of linear maps from V to V.

What does it mean for a Lie subalgebra L of $\mathrm{End}(V)$ to be (i) abelian, (ii) nilpotent or (iii) soluble?

Show that if L is nilpotent and L_1 is a maximal proper Lie subalgebra of L then L_1 is an ideal of L. Give an example where L is soluble with a maximal Lie subalgebra L_1 which is not an ideal.

Define what it means for a linear map $D : L \longrightarrow L$ to be a derivation of L. What does it mean for D to be an inner derivation?

Now suppose L is non-zero and nilpotent. Show that there is a derivation of L that is not inner. Is the same necessarily true if 'nilpotent' is replaced by 'soluble'?

2

Let L be a finite dimensional complex Lie algebra. What does it mean for L to be semisimple?

Now suppose that L is semisimple. Define the Killing form B_L and show that it is non-degenerate. [You may assume Cartan's solubility criterion if clearly stated.]

Define what it means for an abelian Lie subalgebra H to be a Cartan subalgebra of L.

Let H_1 be an abelian Lie subalgebra of L. Does H_1 necessarily lie in some Cartan subalgebra H of L? Justify your answer.

Now let H be a Cartan subalgebra of L. Show that the restriction of B_L to H is non-degenerate. [You may assume that H is the centraliser of some $h \in H$.]

Describe the Cartan decomposition of L with respect to H, explaining why it exists.

3

What is meant by a finite dimensional Lie algebra representation of a complex Lie algebra L? What does it mean for such a representation to be irreducible?

Define the complex Lie algebra sl_2, and describe its irreducible finite dimensional representations. [You do not need to prove that your list is complete, but you should establish irreducibility in each case.] Deduce that sl_2 is a simple Lie algebra.

Let L be a semisimple finite dimensional complex Lie algebra with Cartan subalgebra H. Let α be a root of L (with respect to H) with associated root space L_α. Let U_α be the Lie subalgebra of L generated by L_α and $L_{-\alpha}$. Show that there is a copy of sl_2 in U_α.

By using the representation theory of sl_2, or otherwise, show that each root space L_α is one dimensional. [You may assume that any non-zero finite dimensional representation of sl_2 is the direct sum of irreducible ones.]

Let α and β be roots, and suppose that $\beta \neq m\alpha$ for any integer m. The α-string through β is the longest arithmetic progression of the form $\beta - q\alpha, \ldots, \beta, \ldots, \beta + p\alpha$ (for non-negative integers p and q) all of whose elements are roots. Show that the sum of the root spaces associated with the elements of the α-string form a simple sl_2-module and that for $x \in [L_\alpha, L_{-\alpha}]$, the equation $\beta(x) = (q-p)\alpha(x)/2$ holds.

4

Let R be a finite dimensional associative algebra over an algebraically closed field k.

Define what is meant by the Jacobson radical J of R.

State the Artin-Wedderburn theorem describing the structure of R/J.

Show that J is a nilpotent ideal.

For $1 \leqslant i \leqslant n$, let P_i be an indecomposable right R-module such that P_i/P_iJ is a simple right R module S_i. Suppose that S_i is not isomorphic to S_j if $i \neq j$.

Show that $\mathrm{End}_R(P_i)$ is local.

Let M be the direct sum of the modules P_i (with multiplicity one). Show that there is a surjective ring homomorphism $\theta : \mathrm{End}_R(M) \longrightarrow \mathrm{End}_R(M/MJ)$. Show also that ker θ is a nilpotent ideal of $\mathrm{End}_R(M)$. Deduce that $\mathrm{End}_R(M)$ is a basic algebra with Jacobson radical equal to ker θ.

5

What is meant by a simply-laced positive definite Coxeter graph?

State the classification of such graphs.

Let Q be the quiver with two vertices with a single arrow from vertex 1 to vertex 2.

Describe the root system associated with the underlying Coxeter graph of this quiver, choosing a base of simple roots. How many possible choices are there? Describe the Weyl group of this root system. Define what is meant by a Coxeter element of this Weyl group. How many are there, and what is their order?

Let k be an algebraically closed field. Using this quiver Q as an example, explain how to find the finitely many indecomposable representations of a quiver whose underlying graph is a simply-laced positive definite Coxeter graph.

END OF PAPER