UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 9 June, 2021 12:00 pm to 3:00 pm

PAPER 101

COMMUTATIVE ALGEBRA

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

*Attempt no more than **FOUR** questions.*
*There are **SIX** questions in total.*
The questions carry equal weight.
All rings are assumed to be commutative with a 1.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that the following three conditions are equivalent for an R-module M: (i) every submodule of M is finitely-generated; (ii) the ascending chain condition holds (iii) the maximal condition on submodules holds.

(b) Show that if M is a module and N is a submodule then:

(i) M is finitely-generated if N and M/N are finitely-generated.

(ii) M is noetherian if and only if N and M/N are noetherian.

(iii) The modules $M_1, M_2, \ldots, M_r$ are noetherian if and only if their direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_r$ is noetherian.

Let $I_1, \ldots, I_r$ be ideals of R such that each R/I_j is a noetherian ring. Deduce that $\bigoplus R/I_j$ is a noetherian R-module, and that, if $\bigcap I_j = 0$, then R is also a noetherian ring.

(c) Let $R[X]$ be the ring of polynomials in one indeterminate over the ring R. For any R-module M, let $M[X]$ denote the set of all polynomials in X with coefficients in M, that is to say, expressions of the form

$$m_0 + m_1 X + \cdots + m_r X^r$$

for $m_i \in M$. Define the product of an element of $R[X]$ and an element of $M[X]$ such that $M[X]$ becomes an $R[X]$-module (you are not asked to prove all the details).

Assume that M is a noetherian R-module. By adapting the proof of the Hilbert basis theorem, prove that $M[X]$ is a noetherian $R[X]$-module.

2

(a) Let the subset S of R be multiplicatively closed. Explain briefly the construction which makes $S^{-1}R$ into a ring.

(i) Show that $S^{-1}R = \{0\}$ if and only if S contains a nilpotent element.

(ii) Let A and B be rings. Let $R = A \times B$. Let $S = \{(1,1),(1,0)\}$. Prove that $A = S^{-1}R$.

(iii) Find all intermediate rings $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$, and describe each R as a localisation of $\mathbb{Z}$. [Hint: prove first that $\mathbb{Z}[2/3] = S^{-1}\mathbb{Z}$ where $S = \{3^i : i \geqslant 0\}$.]

(iv) Let R be an integral domain, K its field of fractions, L a finite extension of K and $\bar{R}$ the integral closure of R in L. Show that every element of L can be expressed as a fraction b/a where $b \in \bar{R}$ and $a \in R$.

(b) (i) Let the subset S of R be multiplicatively closed. Show that there is a one-to-one correspondence between prime ideals of $S^{-1}R$ and prime ideals of R which do not meet S.

(ii) Let R be a finitely-generated k-algebra. Prove the Nullstellensatz, namely that $N(R) = J(R)$ (in the usual notation). [You may assume the so-called weak Nullstellensatz provided you state it clearly.]

3

(a) Briefly describe the construction of the *tensor product* $M \otimes_R N$ of two R-modules M and N. Proofs of assertions are not required.

(i) If I is an ideal of R and M an R-module, show that $(R/I) \otimes_R M$ is isomorphic to M/IM. [Hint: You may wish to consider tensoring the exact sequence $0 \to I \to R \to R/I \to 0$ with M.]

(ii) Assume additionally that R is a local ring, with unique maximal ideal $\mathfrak{m}$, say. Let M and N be finitely-generated R-modules. Prove that if $M \otimes_R N = 0$, then $M = 0$ or $N = 0$. [Hint: if k is the residue field observe that $M_k := k \otimes_R M \cong M/\mathfrak{m}M$ by (i) and then apply Nakayama's lemma when $M_k = 0$.]

(b) What does it mean to say that an R-module is *flat*?

Show that free R-modules are flat. [You may assume the result that if M_j $(j \in J)$ is a family of R-modules and M is their direct sum, then M is flat if and only if each M_j is flat.]

Deduce that if $R[X]$ is the ring of polynomials in one indeterminate over R then $R[X]$ is a flat R-algebra. [Recall that an R-algebra A is a ring A together with a ring homomorphism $R \to A$; we say that A is *flat* if it is flat when considered as an R-module by restriction of scalars.]

4

Let B be a ring and A a subring of B.

(a) What does it mean to say that an element x of B is *integral* over A?

(i) Show that the set C of elements of B which are integral over A is a subring of B containing A.

Define the *integral closure* of A in B. What does it mean to say that (1) A is *integrally closed* in B and (2) B is *integral* over A.

(ii) Let G be a finite group of automorphisms of a ring A and let A^G denote the subring of G-invariants, that is, $A^G = \{x \in A : \sigma(x) = x \text{ for all } \sigma \in G\}$. Prove that A is integral over A^G.

(b) Assume that B is integral over A.

(i) State and prove the going-up theorem (the lying-over theorem may be assumed, if stated clearly).

(ii) Show that if $x \in A$ is a unit in B then it is a unit in A. Show also that the (Jacobson) radical of A is the contraction of the Jacobson radical of B, that is, $J(A) = J(B) \cap A$.

(iii) Let $\mathfrak{p}$ be a prime of A. Suppose that B has a unique prime $\mathfrak{p}'$ lying over $\mathfrak{p}$. Quoting carefully any results you use, establish the following three statements (in the usual notation for localisations). Namely, show (1) that $\mathfrak{p}'B_{\mathfrak{p}}$ is the only maximal ideal of $B_{\mathfrak{p}}$ (2) that $B_{\mathfrak{p}'} = B_{\mathfrak{p}}$, and (3) that $B_{\mathfrak{p}'}$ is integral over $A_{\mathfrak{p}}$.

5

(a) Let $A = \bigoplus_{n \geqslant 0} A_n$ be a noetherian graded ring.

(i) Show that A_0 (the zeroth graded piece) is a noetherian ring, and A is finitely-generated as an A_0-algebra.

(ii) Suppose A is generated by elements $x_1, \ldots, x_s$, which we may take to be homogeneous, of degrees $k_1, \ldots, k_s$ (all strictly positive). Suppose M is a finitely-generated graded A-module. Define the *Poincaré series* $P(M,t)$ of M (with respect to some additive function λ with values in $\mathbb{Z}$). State and prove the Hilbert–Serre theorem about P.

(iii) Denote the order of the pole of $P(M,t)$ at $t = 1$ by d. Deduce that if each $k_i = 1$, then for all sufficiently large n, $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree $d - 1$. You may assume the degree of the zero polynomial is -1.

(b) (i) Let R be a ring. Define the *Krull dimension*, $\dim R$, of R. Suppose now that A is an affine k-algebra (k a field) which is an integral domain with field of fractions L. Define the *transcendence degree* $\text{tr.deg}_k L$. Assuming Noether's normalisation lemma, show that $\dim A$ and the transcendence degree are equal.

(ii) Let R be an integral domain of (finite) dimension r, and $\mathfrak{p}$ a non-zero prime. Prove that $\dim(R/\mathfrak{p}) < r$.

(iii) Let A be a subring of B such that B is integral over A. Show that $\dim A = \dim B$. [Hint: you may assume the going-up theorem and the incomparability theorem.]

6

(a) Define the *height* of a prime ideal of R. If R is noetherian, state Krull's Principal Ideal Theorem (the *Hauptidealsatz*) for R. Deduce that if I is a proper ideal generated by n elements and $\mathfrak{p}$ is a minimal prime ideal over I, then the height of $\mathfrak{p}$ is at most n.

(i) Let R be a noetherian local ring with principal prime ideal $\mathfrak{p}$ of height at least 1. Prove that R is an integral domain.

(ii) Let k be a field, and consider $P = k[[X]]$, the formal power series ring in one variable, so that every non-zero ideal has the form (X^n) for some n and $\mathfrak{m} = (X)$ is the unique maximal ideal. Let $R = P \times P$. Prove that R is noetherian and has a finite number of maximal ideals. [You may assume that the primes of R are of the form $\mathfrak{q} \times P$ or $P \times \mathfrak{q}$ where $\mathfrak{q}$ is a prime of P.] Show also that R contains a principal prime $\mathfrak{p}$ of height 1, but that R is not an integral domain.

(b) Let R be a noetherian ring and M a finitely-generated R-module.

For an ideal I of R, what is a *stable I-filtration* (M_n) of M? Define the *Rees ring*, R^*. If (M_n) is an I-filtration of M, let $M^* = \bigoplus_{n \geqslant 0} M_n$. Explain why M^* is a graded R^*-module.

(i) Assuming general facts about graded rings and modules that you need, show that the following are equivalent:

(1) M^* is a finitely-generated R^*-module.

(2) The filtration (M_n) is stable.

If M' is a submodule of M, deduce the Artin-Rees lemma in the form that there exists an integer ℓ such that

$$(I^n M) \cap M' = I^{n-\ell}((I^\ell M) \cap M')$$

for all $n \geqslant \ell$.

(ii) Using the Artin–Rees lemma, derive Krull's intersection theorem in the form that if I is an ideal and $N = \bigcap_{n \geqslant 0} I^n M$, then there exists $x \in I$ such that $(1 + x)N = 0$.

END OF PAPER