UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2019 9:00 am to 11:00 am

PAPER 350

BAYESIAN INVERSE PROBLEMS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider the linear measurement model

$$m = Au + \eta,$$

where $u \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^k$ are assumed to be independent random variables, with probability measures that are absolutely continuous with respect to Lebesgue measure, and $A \in \mathbb{R}^{k\times d}$ is a known matrix. Formulate the posterior probability density function that gives a solution to the inverse problem "given a measurement m approximate u".

(b) Assume that the measurement is given by $m = \langle A, u\rangle + \eta$ where $u \in \mathbb{R}^2$, $\eta \in \mathbb{R}$ and $A = (2, 1) \in \mathbb{R}^2$. Assume that $\eta \sim \mathcal{N}(0, \delta^2)$, with noise level $\delta > 0$, and $u \sim \mathcal{N}(0, I_2)$. Derive the posterior distribution and calculate the posterior mean and covariance. Does the prior play any role in the small noise limit $\delta \to 0$? Is the uncertainty in the solution same to all directions? Explain.

(c) Let μ and μ' denote two probability measures that are absolutely continuous with respect to a third probability measure ν. The Hellinger distance between between μ and μ' is defined as

$$d_{Hell}(\mu, \mu') = \sqrt{\frac{1}{2}\int \left(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}}\right)^2 d\nu}.$$

Let $\mu_1 = \mathcal{N}(\theta_1, \sigma_1^2)$ and $\mu_2 = \mathcal{N}(\theta_2, \sigma_2^2)$ be two Gaussian measures on $\mathbb{R}$. The squared Hellinger distance between μ_1 and μ_2 is

$$d_{Hell}(\mu_1, \mu_2)^2 = 1 - \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}} \exp\left(-\frac{(\theta_1 - \theta_2)^2}{4(\sigma_1^2 + \sigma_2^2)}\right).$$

Prove the above in the special case $\sigma_1 = \sigma_2 = 1$. Show that

$$d_{Hell}(\mu_1, \mu_2)^2 \leqslant C_{\sigma_1,\sigma_2}\big((\sigma_1 - \sigma_2)^2 + (\theta_1 - \theta_2)^2\big),$$

where $C_{\sigma_1,\sigma_2} > 0$ is a constant depending on σ_1 and σ_2. [Hint: $e^{-x} \geqslant (1-x)$.]

2 Assume the linear measurement model

$$m = Au + \eta,$$

where $u \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^k$ are independent random variables, with probability measures that are absolutely continuous with respect to Lebesgue measure, and $A \in \mathbb{R}^{k \times d}$ is a known matrix.

(a) Assume that the posterior density $\pi^m(u) = \pi(u \,|\, m)$ exists. Define the *conditional mean (CM) estimator* and *maximum a posterior (MAP) estimator*. Let $u \sim \mathcal{N}(0, \Sigma)$, where Σ is a positive definite matrix, and $\eta \sim \mathcal{N}(0, I)$. State the CM estimator for u given a measurement m. [Either of the forms given in the lecture notes is accepted].

(b) Let $u \in \mathbb{R}^d$ represent a pixel image, where the j^{th} component u_j represents the intensity of the j^{th} pixel. Assume that the prior knowledge is that the picture contains small and well localised objects with the background having intensity close to zero. Give an example of an appropriate prior density. Assume that random draws from a uniform distribution $\mathcal{U}([0,1])$ are available. Show how to draw samples from the distribution you chose and justify your method.

(c) Assume that the null space of A is zero, and $u \sim \mathcal{U}$, where $\mathcal{U}$ is an uninformative and improper prior with constant density on $\mathbb{R}^d$, that is, $\pi(u) = c > 0$. Furthermore, assume that $\eta \,|\, \delta \sim \mathcal{N}(0, \delta^2 I)$, where $\delta > 0$ is unknown. The noise amplitude is modelled by assuming $1/\delta^2 = \gamma \sim \Gamma(\alpha, \beta)$, where $\alpha, \beta > 0$, and $\Gamma(\alpha, \beta)$ is the Gamma distribution, with the density

$$\pi_h(\gamma) \propto \gamma^{\alpha - 1} \exp(-\beta\gamma).$$

Write down the posterior distribution $\pi^m(u, \gamma)$ and the densities of $u \,|\, \gamma, m$ and $\gamma \,|\, u, m$. Give the MAP estimators for u and γ.

3 (a) Define the *Cameron–Martin space* and *Cameron–Martin theorem* both for a Gaussian measure $\mu = \mathcal{N}(0, \Sigma)$, where Σ is a self-adjoint, positive definite trace-class operator on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$.

(b) Assume measurement model $m = Au + \eta$, where $A = K^*K$, with $K : H^t(\mathbb{R}^2) \to H^{t+2}(\mathbb{R}^2)$ for any $t \in \mathbb{R}$, and K^* is such that $\langle K^*u, v\rangle_{L^2} = \langle u, Kv\rangle_{L^2}$. The unknown u is assumed to follow a Gaussian prior Π, for which $\Pi(L^2(\mathbb{R}^2)) = 1$. Assume white Gaussian noise model, that is, $\eta \sim P_0 = \mathcal{N}(0, I)$. Define the posterior Π^m using Bayes' theorem. Show that it is well defined and justify why we can use Bayes' theorem. [*You may use the properties of white noise and the fact that the potential is ν_0-measurable for $\nu_0(du, dm) = \Pi(du)P_0(dm)$ without proof.*]

(c) Let μ and μ' be two probability measures on a separable Banach space X. Let $(Y, \|\cdot\|)$ be a separable Banach space and assume that $f : X \to Y$ is measurable and has second moments with respect to both μ and μ'. Show that

$$\|\mathbb{E}^{\mu}(f) - \mathbb{E}^{\mu'}(f)\| \leqslant 2\Big(\mathbb{E}^{\mu}\|f\|^2 + \mathbb{E}^{\mu'}\|f\|^2\Big)^{\frac{1}{2}} d_{Hell}(\mu, \mu'),$$

where the Hellinger distance d_{Hell} is defined as in question 1 part (c).

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